

AN INVARIANT FOR MINIMUM TRIANGLE-FREE GRAPHS

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ABSTRACT. We study the number of edges, $e(G)$, in triangle-free graphs with a prescribed number of vertices, $n(G)$, independence number, $\alpha(G)$, and number of cycles of length four, $N(C_4; G)$. We in particular show that

$$3e(G) - 17n(G) + 35\alpha(G) + N(C_4; G) \geq 0$$

for all triangle-free graphs G . We also characterise the graphs that satisfy this inequality with equality.

1. INTRODUCTION

1.1. Background. The (*minimum*) *edge numbers*, $e(3, k, n)$, are defined as the minimum number of edges in a triangle-free graph on n vertices without an independent set of size k . These numbers, and constructions of some related graphs, have successfully been used to compute, or bound, the classical two-colour Ramsey numbers $R(3, \ell)$. In particular for $\ell = 6$ by Kalbfleisch [8], for $\ell = 7$ by Graver and Yackel [5] and for $\ell = 9$ by Grinstead and Roberts [6]. Among the useful upper bounds on the Ramsey numbers $R(3, \ell)$ that have been obtained by these considerations are those of Radziszowski and Kreher (e.g. [9]).

In particular Radziszowski and Kreher proved, in [9], that $e(3, k+1, n) \geq 6n - 13k$ for all non-negative integers n and k . One may differently phrase that result by saying that $t(G) := e(G) - 6n(G) + 13\alpha(G) \geq 0$ for all triangle-free simple graphs $G = (V, E)$ where $e(G) = |E|$ denotes the number of edges, $n(G) = |V|$ the number of vertices and $\alpha(G)$ the independence number of G . Moreover the triangle-free graphs G for which $t(G) = 0$ have been classified in part by Radziszowski and Kreher in [9] and completely by Backelin in [2]. The invariant t is just one in a series of invariants of a similar kind, all of which give bounds on the edge-numbers and for which there is a classification of the triangle-free graphs that satisfy them with equality. In particular we have $e(G) \geq 0$, $e(G) - n(G) + \alpha(G) \geq 0$, $e(G) - 3n(G) + 5\alpha(G) \geq 0$ and $e(G) - 5n(G) + 10\alpha(G) \geq 0$, with full classification of graphs for which we have equality (see, e.g. [9]).

In this paper we consider a related invariant, $\nu(G)$, which we define as

$$\nu(G) = 3e(G) - 17n(G) + 35\alpha(G) + N(C_4; G),$$

where $N(C_k; G)$ denotes the number of cycles of length k in G . We will, in particular, show that $\nu(G) \geq 0$ for all triangle-free graphs G (see Theorem 1 in Section 1.2).

This affirmatively answers a question first considered in [1]. We also give a classification of the graphs that satisfy this inequality with equality. We will see that this bound is tight since there are (infinitely many) triangle-free graphs G for which $\nu(G) = 0$. Interestingly, these graphs seem to be closely related to those for which $t(G) = 0$. In particular there are infinitely many triangle-free graphs for which $t(G) = \nu(G) = 0$.

A specialisation of the bound $\nu(G) \geq 0$ in Theorem 1. Is the bound $e(C_{\leq 4}, k+1, n) \geq \frac{17}{3}n - \frac{35}{3}k$, where $e(C_{\leq 4}, k+1, n)$ is the number of edges in a graph containing no cycle of length at most four on n vertices without an independent set of size $k+1$.

It is not wholly unnatural to involve the quantity $N(C_4; G)$ in the bound. Assume that G is some triangle-free graph. Consider, similarly to Graver and Yackel in [5], the following

$$2e^2(G) := \sum_{v \in V(G)} d(v)^2,$$

where $V(G)$ denotes the set of vertices in G and $d_G(v) = d(G; v)$ denotes the valency of the vertex v in the graph G , where we leave out the subscript if it is clear which graph we are considering from the context. Let $d^2(v)$, called the *second valency* of a vertex v , be defined as $d^2(v) = d^2(G; v) = \sum_{w \in N(v)} d(w)$, where $N_G(v)$ denotes the neighbourhood of v (set of vertices adjacent to v) in G , where again G is dropped from notation if clear from context.

We will by G_v denote the induced subgraph of G obtained by removing v and all its neighbours. We let $N(C_k; G, S)$ denote, where $S \subseteq V(G)$ is a subset of vertices of G , the number of cycles of length k in G that contains at least one vertex from S . If $S = \{v\}$ we will omit the use of set parentheses and write $N(C_k; G, v)$ instead of $N(C_k; G, \{v\})$.

We have that

$$e^2(G) - e^2(G_v) = \sum_{w \in N(v)} \left(d^2(w) + \binom{d(w)}{2} \right) - \binom{d(v)}{2} - (N(C_4; G) - N(C_4; G, v)).$$

Differences of these kinds are quite essential to the methods of Graver and Yackel and the quantities involved in such differences might therefore be interesting to study in relation to graphs with low edge numbers.

1.2. Graphs with ν -value zero and the main theorem. We already know of some triangle-free graphs G such that $\nu(G) = 0$. We will here describe all graphs with ν -value zero. That these are indeed all such graphs will be demonstrated in the conclusion of this article.

We need to define the following class of graphs (which appears in [2] and [1] as *chains* denoted by Ch_k , in [9] as F_k and in [7] as H_k). These graphs will also play an important role in our proofs.

Definition 1. Let Ch_2 be a cycle of length five. We recursively define Ch_{k+1} for $k \geq 2$. Let $x \in V(Ch_k)$ be some bivalent vertex. Let $V(Ch_{k+1}) = V(Ch_k) \cup \{v, w_1, w_2\}$ and $E(Ch_{k+1}) = E(Ch_k) \cup \{vw_1, vw_2, w_1x\} \cup \{w_2y; y \in N(x)\}$.

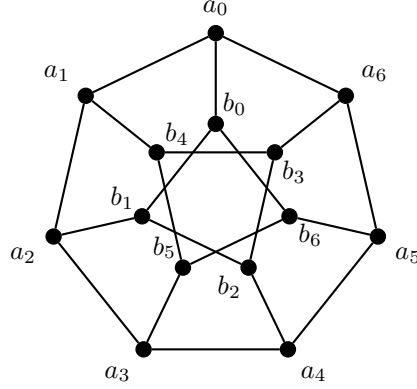
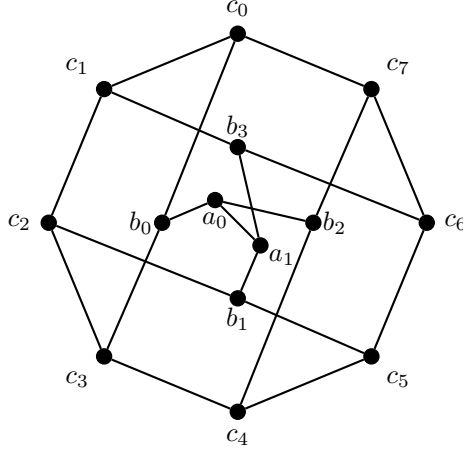
It is easy to verify that Ch_k is then well-defined for $k \geq 2$, i.e. up to isomorphism the result does not depend on the choice of bivalent vertex in the recursive construction. It is also easy to check that $n(Ch_k) = 3k - 1$, $e(Ch_k) = 5k - 5$, $\alpha(Ch_k) = k$ and $N(C_4; Ch_k) = k - 2$. Hence, $\nu(Ch_k) = 0$ for all $k \geq 2$.

There are two connected 3-regular graphs with ν -value 0. These have been characterised in [1]. Using the same notation as there we define the graphs $(2C_7)_{2i}$ and W_5 as follows. Let $V((2C_7)_{2i}) = \{a_0, a_1, \dots, a_6\} \cup \{b_0, b_1, \dots, b_6\}$ and the edges of $(2C_7)_{2i}$ be such that both a_0, a_1, \dots, a_6 and b_0, b_1, \dots, b_6 form cycles of length seven in $(2C_7)_{2i}$. Connect these two cycles by adding an edge $b_i a_{2i}$ for all $i \in \{0, 1, \dots, 6\}$, taking indices modulo 7.

This graph is also known as a generalised Petersen graph, variously denoted $GP(7, 2)$ or $P(7, 2)$.

Let $V(W_5) = \{a_0, a_1\} \cup \{b_0, \dots, b_4\} \cup \{c_0, \dots, c_7\}$ and the edges of W_5 be such that $a_0 a_1$ are adjacent, b_0, \dots, b_4 are independent and c_0, \dots, c_7 form a cycle of length eight. Add edges $b_i a_i$ for $i \in \{1, 2, 3, 4\}$ taking a_i -indices modulo 2. Also add edges $b_i c_{2i}$ and $b_i c_{2i+3}$ for $i \in \{1, 2, 3, 4\}$ taking indices modulo 8.

Property 8 will show, after Theorem 1 has been established, that these two 3-regular graphs are the only 3-regular connected graphs with ν -value zero.

FIGURE 1. The graph $(2C_7)_{2i}$.FIGURE 2. The graph W_5 .

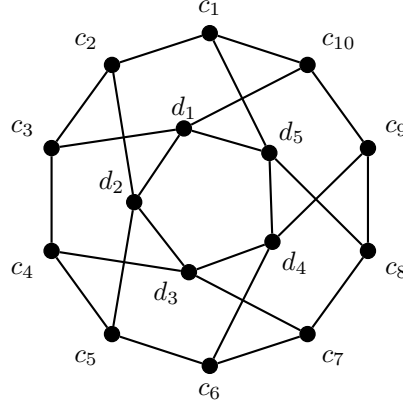
This extends a result in [1] that states that these two graphs are the only two 3-regular graphs with ν -value zero that neither contains cycles of length three nor of length four.

Let BC_k , $k \geq 4$, be a graph consisting of an induced cycle on vertices c_1, c_2, \dots, c_{2k} and one induced cycle on vertices d_1, d_2, \dots, d_k . Connect the cycles by edges $d_i c_{2i-2}$ and $d_i c_{2i+1}$ for $i \in \{1, \dots, k\}$, taking indices modulo $2k$ for c_i s and modulo k for d_i s. The graphs BC_k have been called *bicycles* (in [1] and [2]) or *extended k -chains* (in [7], denoted E_k) and G_k (in [9]).

Note that we have $n(BC_k) = 3k$, $e(BC_k) = 5k$. It is not difficult to show that $\alpha(BC_k) = k$. Moreover, for $k \geq 5$ we have $N(C_4; BC_k) = k$. Hence $\nu(BC_k) = 0$. In the case $k = 4$ we have one “extra” cycle of length four formed by the vertices d_1, d_2, d_3 and d_4 and because of this we have $\nu(Ch_4) = 1$.

Note also that we will not be considering the empty graph, $G = (\emptyset, \emptyset)$. If one were to consider it however, then it is reasonable to define $n(G) = e(G) = \alpha(G) = N(C_4; G) = 0$. Thus also G would be a graph with ν -value zero. All graphs in this paper will however be assumed to be non-empty.

We will prove the following theorem, which is the main result of this paper, in Section 3.2. We now introduce the following notation for the family of graphs with

FIGURE 3. The graph BC_5 .

ν -value zero that have been defined in this section.

$$\mathcal{G} := \{W_5, (2C_7)_{2i}\} \cup \{Ch_k; k \geq 2\} \cup \{BC_k; k \geq 5\}.$$

Theorem 1. If G is a triangle-free graph then $\nu(G) \geq 0$, and if $\nu(G) = 0$, with G connected, then $G \in \mathcal{G}$.

This means that the only triangle-free connected graphs with ν -value zero are those that are those that we have defined in this section. Since ν is linear, in the sense that if $H_1 + H_2$ is the disjoint union of graphs H_1 and H_2 then $\nu(H_1 + H_2) = \nu(H_1) + \nu(H_2)$, we get that it is enough to classify the connected graphs with ν -value zero as in Theorem 1.

1.3. Outline of the proof. To establish Theorem 1 we, in Section 1.4, introduce some preliminary results which we will need later while simultaneously establishing the bulk of the notation that will be used throughout the paper.

In Section 2 we start the systematic study of the properties of the invariant ν for triangle-free graphs. In particular, in Section 2.1 we derive some properties, relating to ν , for graphs G such that all proper subgraphs of G have non-negative ν -value and all proper subgraphs with ν -value zero are in the family of graphs \mathcal{G} . When we later establish Theorem 1 all properties in this section will state general properties that hold for all triangle-free graphs G , since then the condition on the proper subgraphs always holds. The general idea is to derive some local properties for the structure of the neighbourhoods of low valency vertices in subgraphs of G with low ν -value.

Later, in Section 3 we strengthen our assumption, for a contradiction, on G to say that G is a minimal counter-example to satisfying Theorem 1. This means that we assume that G is a triangle-free graph such that $\nu(G) < 0$ or $\nu(G) = 0$ but $G \notin \mathcal{G}$, while $\nu(H) \geq 0$ with equality if and only if $H \in \mathcal{G}$ for all proper subgraphs H of G . Unlike the results from Section 2 the properties we derive in Section 3 do not give us general results for triangle-free graphs relating to ν since we begin by assuming something which we will show leads to a contradiction.

In Section 3.1 we will mimic the work of Radziszowski and Kreher in [9] with the use of a slight modification of their proof mentioned by Backelin in [2]. This section mostly consists of reformulating their results to make them fit into our particular context.

1.4. Preliminaries and notation. We start by stating some preliminary lemmas which we will use later. These are for the most part easy results.

We will let $G - e$ denote, for $e \in E(G)$ the graph obtained by removing the edge e from the graph G . A graph is called *edge-critical* if it has the property that for all edges $e \in E(G)$ its independence number increases as we remove it, i.e. $\alpha(G - e) > \alpha(G)$. This property is also commonly known as the graph being α -critical (for example in [3]). It is easily verified that all the graphs G such that $\nu(G) = 0$ defined in the previous section are edge-critical. In fact, it will follow from Theorem 1 that all triangle-free graphs G such that $\nu(G) < 3$ must be edge-critical.

By abuse of notation we will often say that graphs are equal when they in fact are merely isomorphic. It should be clear from the context that we only need to consider graphs up to isomorphism in such instances. Therefore we either use the notation $H \cong G$ or $H = G$ to say that the graphs H and G are isomorphic.

If S is a set and $k \geq 1$ we will let $\binom{S}{k}$ denote the set of all subsets of S of size k .

A vertex $v \in V(G)$ will be said to be *monovalent* (resp. *bivalent*, *trivalent*, *tetravalent* etc.) if $d(v) = 1$ ($d(v) = 2$, $d(v) = 3$, $d(v) = 4$, etc.) in G .

Lemma 1. (Lemma 2.4 in [2]) If G is an edge-critical triangle-free graph, then $\alpha(G_v) = \alpha(G) - 1$ for all $v \in V(G)$.

Proof. It is obviously true for $v \in V(G)$ such that $d(v) = 0$. Suppose therefore that v is at least monovalent.

We see that $\alpha(G_v) \leq \alpha(G) - 1$ since $S \cup \{v\}$ is an independent set of G for all $v \in V(G)$ and any maximum independent set S of G_v .

If $\alpha(G_v) + 2 \leq \alpha(G)$ then there would be a maximum independent set, S , of size at least $\alpha(G_v) + 3$ in $G - e$, where $e = \{v, w\}$, since G is edge-critical. Then $v \in S$ and thus no G -neighbours of v other than w is in S . Hence $S \setminus \{v, w\}$ is an independent set of size at least $\alpha(G_v) + 1$ in G_v , a contradiction. \square

For a vertex v in a graph G we denote the vertices at distance exactly k from v in G by $N_{k,G}(v)$. If there is no ambiguity for the graph G we will just write $N_k(v)$. For $S \subseteq V(G)$, we shall let $G[S]$ denote the induced subgraph on S and by $G \setminus S$ the induced subgraph $G[V(G) \setminus S]$. If $S = \{v\}$ contains only one element we sometimes omit the usage of set parentheses and write $G \setminus v$ for $G \setminus \{v\}$. If $T \subseteq E(G)$ then we let $G - T$ denote the graph $(V(G), E(G) \setminus T)$ where the edges in T are removed from G . If $T = \{e\}$ we will similarly omit the usage of set parentheses. Note in particular the distinction between $G \setminus e$ and $G - e$. A set of vertices $S \subseteq V(G)$ is said to *destabilise* G if $\alpha(G \setminus S) < \alpha(G)$. If S destabilises G then the set S is called a *destabiliser*. If S is a destabiliser of G such that for all proper subsets $T \subsetneq S$ the set T does not destabilise G , then S is called a *minimal destabiliser*. If G has no destabilisers of size r or less then G is said to be *r-stable*.

We say that a subset $S \subseteq V(G)$ is *connected in* G if $G[S]$ is a connected graph. We will use $V_k(G)$, for $k \geq 0$, to denote the set of all vertices in G of valency exactly k , i.e. $V_k(G) = \{v \in V(G); d(v) = k\}$.

Lemma 2. If G is a triangle-free graph, $v \in V(G)$ and $N_2(v)$ does not destabilise G_v then $\alpha(G) \geq \alpha(G_v) + d(v)$.

Proof. Suppose that $N_2(v)$ does not destabilise G_v . Then there is an independent set, I , of size $\alpha(G_v)$ in G_v such that $I \cap N_2(v) = \emptyset$. Since G is triangle-free we have that $I \cup N(v)$ is an independent set, of size $\alpha(G_v) + d(v)$. \square

Lemma 3. (Lemma 2.2 in [2]) If G is a connected edge-critical triangle-free graph, $v \in V(G)$ and $d(v) \geq 2$ then $N_2(v)$ is a destabiliser of G_v .

Proof. Suppose that $N_2(v)$ did not destabilise G_v . By Lemma 2 there is an independent set, of size $\alpha(G_v) + d(v) \geq \alpha(G_v) + 2$, in H , contradicting Lemma 1. Therefore $N_2(v)$ destabilises G . \square

We will use the following well-known lemma about the minimum valency of edge-critical graphs.

Lemma 4. (see e.g. [4, Prop. 1, Ch. 13]) If G is an edge-critical graph then $\delta(G) \geq 2$ unless $G \cong K_1$ or $G \cong K_2$.

An edge $e \in E(G)$ will be called *redundant* in G if $\alpha(G - e) = \alpha(G)$, otherwise e is said to be *critical*. Note that a graph is edge-critical precisely when it has no redundant edges.

For a set of vertices $W \subseteq V(G)$ we let $N[W]$ denote the *closed neighbourhood* of the vertices in W , which is the set of vertices that are either in W or adjacent to a vertex in W . If $W = \{w\}$ has size one we omit the set parentheses and write $N[w] = N[\{w\}]$. If S is an independent set of vertices we let G_S denote the graph $G[V(G) \setminus N[S]]$, i.e. the graph obtained by removing all the vertices in S , all their neighbours and edges incident to all such vertices. When we have an independent set $S = \{s_1, s_2, \dots, s_k\}$ we will drop the usage of set parentheses in the subscript and write G_{s_1, s_2, \dots, s_k} for $G_{\{s_1, s_2, \dots, s_k\}}$.

Lemma 5. (Lemma 2.6 of [2]) Let G be an edge-critical, connected and triangle-free graph. If $v \in V(G)$ is a bivalent vertex, then G_v is connected.

Proof. Let $\{w_1, w_2\} = N(v)$ and we will write X for $N(w_1) \cup N(w_2) \setminus \{v\}$. Suppose that $G_v = G_1 + G_2$ and set $X_i = X \cap V(G_i)$.

Not both $X_1 \cap N(w_1)$ and $X_2 \cap N(w_2)$ can be empty since G is connected. We may without loss of generality assume that $X_1 \cap N(w_1) \neq \emptyset$.

Suppose that X_1 destabilises G_1 and let S be a maximum independent set of $G - \{w_1, x\}$ where $x \in X_1 \cap N(w_1)$. Then $w_1 \in S$ and therefore $v \notin S$. If $w_2 \in S$ then $|S \cap V(G_1)| \leq \alpha(G_1) - 1$, and if $w_2 \notin S$ then $|S \cap N[v]| = 1$. Hence, in either case we have

$$|S| = |S \cap V(G_1)| + |S \cap V(G_2)| + |S \cap N[v]| \leq \alpha(G_1) + \alpha(G_2) + 1 = \alpha(G),$$

contradicting that G is edge-critical.

Hence X_1 does not destabilise G_1 and analogously X_2 does not destabilise G_2 . But then $\alpha(G) \geq \alpha(G_1) + \alpha(G_2) + 2$, contradicting Lemma 1. \square

We first classify the minimal destabilisers of minimum size in Ch_k -graphs.

Lemma 6. If G is edge-critical, $S \subseteq V(G)$ destabilises G and $v \in V(G) \setminus S$, then $S \cap V(G_v)$ destabilises G_v .

Proof. Otherwise there would be a maximum independent set T of G_v avoiding $S \cap V(G_v)$. $T = \alpha(G_v) = \alpha(G) - 1$, by Lemma 1, and therefore $T \cup \{v\}$ would be a maximum independent set of G avoiding S . This contradicts that S destabilises G . \square

Lemma 7. (Lemma 6.2(b) of [1]) If S destabilises $G = Ch_k$, where $k \geq 2$, then $|S| \geq 3$ with equality if and only if $S = N[v]$ for some bivalent $v \in V(G)$.

Proof. Induction on k . Easily verified for $k = 2, 3$. Let $k \geq 4$ and suppose that the statement holds for all $G = Ch_d$, where $d < k$.

Suppose that $|S| \leq 2$. Since $k \geq 4$ there is some bivalent vertex $v \in V(G) \setminus S$ and therefore, by Lemma 6, $S \cap V(G_v)$ would destabilise $G_v \cong Ch_{k-1}$. This, however, contradicts the inductive assumption. Hence $|S| \geq 3$.

Clearly $N[v]$ destabilises G for all bivalent vertices $v \in V(G)$. Hence it only remains to show that if $|S| = 3$ then $S = N[v]$ for some bivalent vertex v . Suppose, for a contradiction, that $|S| = 3$ but $S \neq N[v]$ for all bivalent vertices $v \in V(G)$.

We then have that $S \cap N[v] \neq \emptyset$ for all bivalent vertices $v \in V(G)$, since otherwise S would destabilise $G_u \cong Ch_{k-1}$ and therefore by the induction hypothesis $S = N_{G_v}[u]$ for some, in G_v , bivalent vertex $u \in V(G_v)$. But u is not bivalent in G since $S \neq N[u]$ and therefore u has distance two from v . Let T be an independent set of size $k-2$ in $G_{v,u}$. Then $T \cup N(v)$ is an independent set in G of size k avoiding S , contradicting that S destabilises G .

Since $|S| = 3$ and there are four bivalent vertices in G there is at least one bivalent vertex that is not in S . Let $v \in V(G)$ be such a vertex. Then $S \cap N(v) \neq \emptyset$ by the above. By induction $S \setminus N[v]$ does not destabilise G_v , which contradicts Lemma 6. \square

For minimal destabilisers of size four we will not completely classify them, but the following lemma tells us that in all but one case they are connected.

Lemma 8. (Lemma 6.2(e) of [1]) If S is a minimal destabiliser of $G = Ch_k$ such that $|S| = 4$ and S is not connected in G , then $k = 3$ and $S = V_2(Ch_k)$.

Proof. It is easy to verify, by checking all possibilities, that the statement holds for $G = Ch_2$ and $G = Ch_3$.

Suppose that $k \geq 4$. Let S be a disconnected destabiliser of $G = Ch_k$ of size four and suppose that $S \not\supseteq N[u]$ for all bivalent vertices $u \in V_2(G)$.

Suppose $V_2(G) = \{s_1, s_2, s_3, s_4\}$, where $s_1s_2, s_3s_4 \in E(G)$. Let u and v denote the neighbours of s_1 and s_2 that are not in $V_2(G)$, respectively. Then $G_{s_1,v} \cong Ch_{k-2}$ and there is an independent set I of size $k-2$ avoiding $\{s_3, s_4\}$, by Lemma 8. Then $I \cup \{u, v\}$ is an independent set of size k in G avoiding $V_2(G)$. Hence $V_2(G)$ does not destabilise G . Therefore there is some bivalent vertex, $v \in V_2(G)$, such that $v \notin S$. Fix such a vertex $v \in V_2(G)$.

By Lemmas 6 and 7 we have that $|S \cap V(G_v)| \geq 3$ and therefore $|S \cap N[v]| = |S| - |S \cap V(G_v)| \leq 4 - 3 = 1$. In fact, $S \cap N[v] = \emptyset$ since otherwise $|S \cap N[v]| = 1$ and therefore $|S \setminus N[v]| = 3$, where $S \setminus N[v]$ destabilises G_v (otherwise add v to an maximum independent set of G_v). Hence $S \setminus N[v] = N_{G_v}[u]$ for some $u \in V_2(G_v)$ by Lemma 8. But $S \not\supseteq N[u]$ for all $u \in V_2(G)$ so u must be trivalent in G . Both neighbours of v in G are adjacent to $N_{G_v}[u]$ and $S \cap N(v) \neq \emptyset$, thus S would be connected.

$S \not\supseteq N_{G_v}[u]$ for all $u \in V_2(G_v)$ since otherwise $u \in V_2(G_v)$ but $u \notin V_2(G)$ because S is minimal so $|S \setminus N[u, v]| \leq 1$ and therefore there would be some independent set I in $G_{v,u}$ of size $k-2$ avoiding S . Because $S \cap N[v] = \emptyset$ we would get that $I \cup N(v)$ would be independent in G of size k , contradicting that S destabilises G .

We will now use these facts to prove, by induction, that every disconnected destabiliser of size four in $G = Ch_k$, $k \geq 4$ contains $N[u]$ for some $u \in V_2(G)$. For $k = 4$ let $v \in V_2(G)$ be such that $v \notin S$ as in the above. Then since $S \cap N[v] = \emptyset$ and $\forall u \in V_2(G_v) : S \not\supseteq N_{G_v}[u]$ we must have that $S = V_2(G_v)$. However, then there is an independent set, I , of size 2 in $G_{v,w} \cong Ch_2 \cong C_5$ avoiding S , where $w \in V_2(G_v)$, since $|S \cap V(G_{v,w})| = 2$ and therefore $S \cap V(G_{v,w})$ does not destabilise $G_{v,w}$. Then $I \cup N(v)$ would be a maximum independent set in G avoiding S , contradicting that S is a destabiliser.

Let $k \geq 5$ and suppose that every disconnected destabiliser of size four in Ch_ℓ , $4 \leq \ell < k$ contains $N_{Ch_\ell}[u]$ for some $u \in V_2(Ch_\ell)$. Suppose also that S is a disconnected destabiliser of size four in $G = Ch_k$. Take $v \in V_2(G)$ such that $v \notin S$ as before. Then $S \cap N[v] = \emptyset$ and $\forall u \in V_2(G_v) : S \not\supseteq N_{G_v}[u]$, thus we get that $S \supseteq N_{G_v}[u] = N[u]$ for some vertex $u \in V_2(G_v) \cap V_2(G)$. \square

Many of the graphs we will study are such that if we remove a vertex v and all its neighbours from G , then we get a Ch_k -graph for some $k \geq 2$. The following lemma tells us that if v is bivalent with second valency six then the local neighbourhood of v in G has a certain structure.

Lemma 9. Let G be such that $G_v = Ch_k$ for some $k \geq 2$ and some $v \in V(G)$ such that $d(v) = 2$ and $d^2(v) = 6$. If $N_2(v)$ destabilises G_v then either $N_2(v)$ contains a pair of adjacent bivalent vertices of G_v or $N(C_4; G, N(v)) \geq 2$.

Proof. Suppose $N_2(v)$ destabilises $G_v = Ch_k$. By Lemmas 7 and 8 in addition to the fact $|N_2(v)| \leq 4$ we have that at least one of the following three statements must hold:

- (i) $N_2(v) \supseteq N_{G_v}[u]$ for some $u \in V_2(G_v)$, or
- (ii) $k = 3$ and $N_2(v) = V_2(G_v)$, or
- (iii) $|N_2(v)| = 4$ and $N_2(v)$ is connected.

In cases (i) and (ii) we get that $N_2(v)$ does contain a pair of adjacent bivalent vertices of G_v . In case (iii) it is easily checked that $N(C_4; G, N(v)) \geq 2$ since the only three connected triangle-free graphs on four vertices (that $N_2(v)$ possibly induce in G_v) are P_4 , C_4 and $K_{1,3}$. \square

Lemma 10. (Lemma 2.10(b) in [1]) If $e \in E(G)$ is redundant in G and $x \in V(G) \setminus N[e]$, then either

- (i) e is redundant in G_x , too, or
- (ii) $\alpha(G_x) \leq \alpha(G) - 2$.

Proof. Suppose that $\alpha(G_x - e) = \alpha(G_x) + 1 = \alpha(G)$. Then if I is a maximum independent set in $G_x - e$ we get that $I \cup \{x\}$ is an independent set of size $\alpha(G) + 1$ in $G - e$, contradicting that e is redundant.

Hence, either $\alpha(G_x - e) \neq \alpha(G_x) + 1$, in which case e is redundant in G_x as well, or $\alpha(G_x) + 1 \neq \alpha(G)$, in which case $\alpha(G_x) \leq \alpha(G) - 2$. \square

For subsets of vertices $A, B \subseteq V(G)$ we write $E_G(A, B)$ for the set of edges with one endpoint in A and the other endpoint in B , i.e. $E_G(A, B) = E(G) \cap \{\{a, b\}; a \in A, b \in B\}$. The cardinality of this set will be denoted by $e_G(A, B)$. If the graph G is clear from context we will drop the subscript from the notation. We will also sometimes abuse the notation by writing $E_G(H_1, H_2)$ for $E_G(V(H_1), V(H_2))$ where H_1 and H_2 are two subgraphs of G .

If S is a collection of sets then we let $\bigcup S$ denote the union of all the sets in S , i.e. $\bigcup S = \bigcup_{X \in S} X$. In particular, if H is an induced subgraph of G then $\bigcup\{e \cap V(H); e \in E(H, G \setminus H)\}$ is the set of vertices of H that are adjacent, in G , to some vertex outside H .

Lemma 11. If H is an induced subgraph of G and $M = \bigcup\{e \cap V(H); e \in E(H, G \setminus H)\}$ does not destabilise H . Then every edge in $E(H, G \setminus H)$ is redundant.

Proof. Suppose that $e \in E(H, G \setminus H)$ were not redundant, then $\alpha(G - e) = \alpha(G) + 1$. Let S be a maximum independent set of $G - e$. $S' = S \cap V(H)$ is independent in H . Since M does not destabilise H there is a maximum independent set S'' of H such that $S'' \cap M = \emptyset$. It follows that $(S \setminus S') \cup S''$ is independent, in $G - e$, of size at least $\alpha(G) + 1$. But since $(S \setminus S') \cup S''$ avoids $e \cap V(H)$ the set $(S \setminus S') \cup S''$ would also be independent in G , a contradiction. \square

As an immediate consequence of this lemma we have the following corollary.

Corollary 1. If H is an induced subgraph of G , H is r -stable and $e(H, G \setminus H) \leq r$ then all edges in $E(H, G \setminus H)$ are redundant.

In addition to Lemmas 7 and 8 which give us a description of the small minimal destabilisers in Ch_k -graphs we will also encounter situations where we have graphs such as Ch_k with an extra edge added between two (in Ch_k) bivalent vertices. We will want to say something about the destabilisers of such a graph as well. In particular we will make use of the following lemma.

Lemma 12. Let $G = Ch_k + e$ where $e = \{\alpha, \beta\} \notin E(Ch_k)$, $k \geq 3$. Moreover suppose that both α and β are bivalent in Ch_k . Let $S \subseteq V(G)$, $|S| = 3$, be such that S contains at least two of the bivalent vertices of Ch_k and if $|S \cap V_2(Ch_k)| = 2$, then $S \cap V_2(Ch_k)$ is an independent set. Then G is not destabilised by S .

Proof. If $S \cap \{\alpha, \beta\} \neq \emptyset$ let $S' := S$. By assumption $S' \neq N_{Ch_k}[v]$ for all $v \in V_2(Ch_k)$ and therefore S' does not destabilise Ch_k by Lemma 7.

Otherwise $|S \cap V_2(Ch_k)| = 2$. The vertex in $S \setminus V_2(Ch_k)$ is adjacent to at most one bivalent vertex in Ch_k and therefore either $S \cup \{\alpha\}$ or $S \cup \{\beta\}$ is a disconnected set in Ch_k of size four not containing all bivalent vertices of Ch_k . By Lemmas 7 and 8 this set is not a destabiliser of Ch_k . In this case define $S' := S \cup \{z\}$ where $z \in \{\alpha, \beta\}$ and S' does not destabilise Ch_k .

There is some independent set I of size $\alpha(Ch_k) = k$ in Ch_k such that $I \cap S' = \emptyset$. But then I contains at most one endpoint of $\{\alpha, \beta\}$ and therefore I is also independent in G . Moreover $\alpha(G) = \alpha(Ch_k) = k$ since there are maximum independent sets in Ch_k avoiding α , for instance.

Hence I is a maximum independent set in G such that $S' \cap I = \emptyset$, so $S \subseteq S'$ does not destabilise G . \square

Let \mathcal{H} be a set of graphs. A graph G is called \mathcal{H} -avoiding if G has no subgraph that is isomorphic to any of the graphs in \mathcal{H} . The set of cycle graphs of lengths $3, 4, \dots, k$ will be denoted by $C_{\leq k}$. The length of the shortest cycle in G is called the *girth* of G . Note that G is $C_{\leq k}$ -avoiding if and only if G has girth at least $k+1$.

The *distance* between two vertices $u, v \in V(G)$ is denoted $\text{dist}_G(u, v)$ and is defined to be the least number of edges in a path from u to v , or infinity if there is no such path. If the graph is clear from context we will drop the subscript from the notation.

Lemma 13. If G is a $C_{\leq 4}$ -avoiding graph which contains a k -cycle $C = c_1, c_2, \dots, c_k$, then for all $v \in V(G) \setminus C$ we have $|N(v) \cap C| \leq \lfloor \frac{k}{3} \rfloor$.

Proof. Let $H := G[C]$ be the induced graph on C . Since G is $C_{\leq 4}$ -avoiding we have that $\forall u_1, u_2 \in N(v) : \text{dist}_{G \setminus v}(u_1, u_2) \geq 3$. Hence, a fortiori, $\text{dist}_H(u_1, u_2) \geq 3$ for all $u_1, u_2 \in N(v) \cap C$ so at most a third of the vertices of C can be in the neighbourhood of v . The lemma follows. \square

In particular the previous lemma gives us that if we have a cycle of length five in a subgraph of G then any vertex outside the cycle can be adjacent to at most one vertex in the cycle. This fact will be used frequently in what follows.

2. THE INVARIANT AND BASIC PROPERTIES

We will prove that ν is a non-negative invariant for all triangle-free graphs G . Let $\mathcal{C}(G)$ denote the set of connected components of the graph G . Note in particular that $\nu(G) = \sum_{C \in \mathcal{C}(G)} \nu(C)$.

Property 1. Let G be a triangle-free graph, then

$$\forall v \in V(G) : \nu(G_v) \leq \nu(G) - 3d^2(v) + 17d(v) - 18 - N(C_4; G, N(v)).$$

Proof. Note that $n(G_v) = n(G) - d(v) - 1$ and $e(G_v) = e(G) - d^2(v)$ (since G is triangle-free). Also we have that $\alpha(G_v) \leq \alpha(G) - 1$ since any maximum independent set in G must either contain v or a vertex in the neighbourhood of v . Also, $N(C_4; G_v) = N(C_4; G) - N(C_4; G, N(v))$ since any cycle of length four through v also goes through one of the neighbours of v .

Therefore, we get that

$$\begin{aligned}\nu(G_v) &\leq \nu(G) - 3d^2(v) + 17d(v) + 17 - 35 - N(C_4; G, N(v)) \\ &= \nu(G) - 3d^2(v) + 17d(v) - 18 - N(C_4; G, N(v)).\end{aligned}$$

□

We will let $H \leq G$ denote that H is a subgraph of G and $H < G$ will denote that H is a proper subgraph of G (i.e. that $H \leq G$ but H is not equal to G).

We will often want to determine a bound for $\nu(G_{s_1, s_2, \dots, s_k})$ in $\nu(G)$. We will then use the notation

$$\nu(G_{s_1, s_2, \dots, s_k}) \leq \nu(G) + d_1 - (c_1) + d_2 - (c_2) + \dots + d_k - (c_k)$$

to indicate that $\nu(G_{s_1, s_2, \dots, s_\ell}) - d_\ell + c_\ell \leq \nu(G_{s_1, s_2, \dots, s_{\ell-1}})$ and that $N(C_4; G_{s_1, s_2, \dots, s_\ell}) + c_\ell \leq N(C_4; G_{s_1, s_2, \dots, s_{\ell-1}})$. Intuitively this may be thought of as saying that when removing s_1 and all its neighbours from G the ν -value increases by at most $d_1 - c_1$ and we remove at least c_1 vertices from the graph. Then we remove s_2 from G_{s_1} and $d_2 - c_2$ gives a bound on the increase in ν -value from G_{s_1} to G_{s_1, s_2} and c_2 indicates the least number of cycles known to be removed. It is sometimes useful to think of G_{s_1, s_2, \dots, s_k} as the vertices, s_1, s_2, \dots, s_k , are being removed in sequence and in each step we keep track of how much the ν -value and the number of cycles of length four changes.

2.1. Properties of a graph for which all subgraphs have non-negative ν -value. From here on we assume that G is a triangle-free graph such that every graph with fewer vertices or the same number of vertices but fewer edges than G has non-negative ν -value, i.e.

$$\begin{aligned}\forall H : n(H) < n(G) \text{ or } n(H) = n(G) \wedge e(H) < e(G) : \\ \text{(A1)} \quad & (1) \nu(H) \geq 0 \text{ and} \\ & (2) \text{ if } \nu(H) = 0, H \text{ connected, then } H \in \mathcal{G}.\end{aligned}$$

Note that this means in particular that for all $H < G$ (all proper subgraphs H of G) we assume that $\nu(H) \geq 0$ and if $\nu(H) = 0$, then $H \in \mathcal{G}$. This is how we will most often use assumption (A1).

We then derive some properties for G and its subgraphs. Note that when Theorem 1 has been established all properties for G that have been derived under assumption (A1) will be shown to hold in general, for all triangle-free graphs G .

Lemma 14. Let $\nu(G) \leq -1$ then G is connected.

Proof. If $\nu(G) \leq -1$ but $G = G' + G''$ for nonempty graphs G' and G'' then we would have that $\nu(G') + \nu(G'') = \nu(G) \leq -1$, thus $\nu(G') \leq -1$ or $\nu(G'') \leq -1$. This would contradict assumption (A1). □

Property 2. If $H \leq G$ and $\nu(H) \leq 2$ then H is edge-critical.

Proof. Suppose that $H \leq G$ and $\nu(H) \leq 2$ but H is not edge-critical. Then there is an edge $e \in E(H)$ such that $\alpha(H - e) = \alpha(H)$. Hence,

$$\nu(H - e) = \nu(H) - 3 \leq -1,$$

which contradicts assumption (A1). □

We let $\delta(H)$ denote the minimum valency of a graph H and $\Delta(H)$ the maximum valency. The complete graph on ℓ vertices is denoted by K_ℓ .

Property 3. If $H \leq G$ and $\nu(H) \leq 17$ then H is 1-stable and $\delta(H) \geq 1$.

Proof. First note that

$$\nu(K_1) = 3 \cdot 0 - 1 \cdot 17 + 35 + 0 = 18.$$

Suppose that $H \leq G$ is such that $\nu(H) \leq 17$ but $\delta(H) = 0$. Let $v \in V(H)$ be a vertex of valency zero. Then we have that

$$\nu(H - v) = \nu(H) - \nu(K_1) \leq 17 - 18 = -1.$$

Moreover, if S were a 1-destabiliser then $\nu(H \setminus S) \leq \nu(H) - 18 \leq -1$. In both situations we would get a contradiction to assumption (A1). \square

Property 4. If $H \leq G$ and $\nu(H) \leq 3$ then $\delta(H) \geq 2$.

Proof. Let $H \leq G$ be such that $\nu(H) \leq 3$. Suppose that $\delta(H) < 2$, then $\delta(H) = 1$ since $\delta(H) = 0$ would contradict Property 3. Let $v \in V(H)$ be a monovalent vertex such that $d^2(v) = \max\{d^2(v); v \in V(H), d(v) = 1\}$. If $d^2(v) = 1$ then v belongs to a K_2 -component of H , which means that $H = H' + K_2$ for some subgraph $H' < G$. But then $3 \geq \nu(H) = \nu(H') + \nu(K_2) = \nu(H') + 4$, which gives us that $\nu(H') \leq -1$ which contradicts assumption (A1).

Hence, $d^2(v) \geq 2$ and therefore we get by Property 1 that

$$\nu(H_v) \leq \nu(H) - 6 + 17 - 18 - N(C_4; H, N(v)) \leq -4,$$

again contradicting assumption (A1). \square

Property 5. If $H \leq G$ and $\nu(H) \leq 7$ then $\delta(H) \leq 4$.

Proof. Suppose that $H \leq G$, $\nu(H) \leq 7$ but $\delta(H) = \delta \geq 5$. Then there is a vertex $v \in V(H)$ of minimum valency $d(v) = \delta$. Such a vertex must have $d^2(v) \geq \delta^2$ and therefore by Property 1

$$\nu(H_v) \leq \nu(H) - 3\delta^2 + 17\delta - 18 - N(C_4; H, N(v)) \leq \nu(H) - 8 \leq -1,$$

since $-3\delta^2 + 17\delta \leq 10$ for $\delta \geq 5$, contradicting assumption (A1). \square

Property 6. If $H \leq G$ and $\nu(H) \leq 3$ then H is 2-stable.

Proof. By Property 4 we get that $\delta(H) \geq 2$. If S is a destabiliser of size 2, then $\nu(H \setminus S) \leq \nu(H) - 9 + 34 - 35 \leq -7$, contradicting assumption (A1). \square

Property 7. If $H \leq G$, $\nu(H) \leq 6$, $C \in \mathcal{C}(H)$ and C is 2-regular, then $C = C_5$.

Proof. Suppose that $H \leq G$ with $\nu(H) \leq 6$ and C is a 2-regular component of H . Then $C = C_m$ for some $m \geq 4$ since G is triangle-free and therefore so is H . We must have that $\nu(C) \leq 6$ since otherwise we would get that $\nu(H - C) = \nu(H) - \nu(C) \leq -1$. But $n(C_m) = e(C_m) = m$ and $\alpha(C_m) \geq \frac{m-1}{2}$, whence $\nu(C_m) \geq 3m - 17m + 35 \left(\frac{m-1}{2}\right) = \frac{7(m-5)}{2}$. However, $\frac{7(m-5)}{2} \geq 7$ for $m \geq 7$ and $\nu(C_6) = 21$. Moreover, $\nu(C_4) = 15$ so the only remaining possibility is that $m = 5$ and $C = C_5$. \square

Inductively, by assumption (A1), we have that if $H < G$ has a 3-regular component, then it is one of the two 3-regular graphs W_5 and $(2C_7)_{2i}$ defined in Section 1.2.

Property 8. If $\nu(G) \leq 0$, $C \in \mathcal{C}(G)$ and C is 3-regular, then $C \in \{(2C_7)_{2i}, W_5\}$.

Proof. The conclusion follows immediately from assumption (A1) if G is not connected. It is therefore enough to consider when G is connected. Suppose that $G \notin \{(2C_7)_{2i}, W_5\}$.

Suppose furthermore that $N(C_4; G) = 0$. By the corollary in [1, p. 202, v. 2015-07-16] we have that $\nu(C) \geq 1$, since C is 3-regular and not one of the two graphs $(2C_7)_{2i}$ or W_5 . But then we would have

$$\nu(G - C) = \nu(G) - \nu(C) \leq -1,$$

contradicting assumption (A1).

Hence, $N(C_4; G) \geq 1$. Let $\{a, b, c, d\}$ be the vertices of a cycle of length four in G . Then $\nu(G_a) \leq \nu(G) + 6 - 1 \leq 5$ and c is at most monovalent in G_a , whence c is monovalent in G_a by Property 3. Since $e(N(a), N(c)) \leq 1$ (otherwise we would have a triangle) we get that $\nu(G_{a,c}) \leq \nu(G_a) - 7 = -2$, contradicting assumption (A1). \square

Property 9. For all $H \leq G$ such that $\nu(H) \leq 6$ either $\delta(H) \geq 2$ or H is edge-critical and $H = H' + K_2$ where $\nu(H') = \nu(H) - 4 \leq 2$ (whence, in particular, $\delta(H') \geq 2$ and H' is edge-critical).

Proof. Let $H \leq G$ be such that $\nu(H) \leq 6$. By Property 4 we have that the assertion is true if $\nu(H) \leq 3$, therefore we may assume that $4 \leq \nu(H) \leq 6$.

If H is not edge-critical, then there is an $e \in E(H)$ such that $\alpha(H - e) = \alpha(H)$, whence $\nu(H - e) \leq 6 - 3 = 3$. Thus by Property 4 we have that $\delta(H - e) \geq 2$, whence also $\delta(H) \geq 2$, as desired.

If on the other hand H is edge-critical, then we may assume that $\delta(H) \leq 1$ since otherwise the assertion is trivially satisfied. This means that H contains a component $C \in \mathcal{C}(H)$ such that $C \cong K_2$ by Lemma 4 and Property 3.

Therefore $H = H' + K_2$ where we get

$$6 \geq \nu(H) = \nu(H') + \nu(K_2) \Rightarrow 2 \geq \nu(H').$$

\square

Property 10. If $H \leq G$ and $\nu(H) \leq 6$ then either H is 2-stable or $H = H' + K_2$ where $V(K_2)$ is the only 2-destabiliser.

Proof. By Property 9 we have that $\delta(H) \geq 2$ or $H = H' + K_2$. If $\delta(H) \geq 2$ then any destabiliser, S , of size 2 would be such that $\nu(H \setminus S) \leq \nu(H) - 9 + 35 - 35 \leq -4$, contradicting assumption (A1).

If $H = H' + K_2$ then $\nu(H') \leq \nu(H) - 4 \leq 2$ and therefore H' is 2-stable by Property 6. Any destabiliser S must contain a destabiliser of one of the components of H . If $|S| = 2$ then S must destabilise the K_2 -component. Hence $S = V(K_2)$. \square

Property 11. If $H \leq G$, $\nu(H) \leq 4$ and $\delta(H) \geq 3$ then H is 3-stable.

Proof. Otherwise let S be a destabilising set of size 3. Since H is triangle-free $e(S) \leq 2$ and therefore $e(H \setminus S) \leq e(H) - 7$. Hence $\nu(H \setminus S) \leq \nu(H) - 21 + 51 - 35 \leq -1$, contradicting assumption (A1). \square

Property 12. If $H \leq G$, $\delta(H) = \delta$ and $v \in V(H)$ is a vertex of valency d then either $N_2(v)$ destabilises H_v or $\nu(H) \geq 3\delta d + 18d - 17$.

Proof. Suppose that $N_2(v)$ did not destabilise H_v , then there would be some independent set, I , of size $\alpha(H_v)$ in H_v such that $I \cap N_2(v) = \emptyset$. Since H is triangle-free we would get that $I \cup N(v)$ is an independent set of size $\alpha(H_v) + d(v)$ in H . Therefore $\nu(H_v) \leq \nu(H) + 17 - 18d - 3\delta d$ since $\alpha(H) \geq \alpha(H_v) + 3$, $n(H) = n(H_v) - (d + 1)$ and $e(H) \geq e(H_v) + \delta d$. The assertion then follows from the assumption (A1) on H_v . \square

Property 13. If $H \leq G$ is such that $\nu(H) \leq 2$ and $C_5 \notin \mathcal{C}(H)$ then for all bivalent vertices $v \in V(H)$ we have that

- (i) $5 \leq d^2(v) \leq 6$ with $d^2(v) = 5$ if $\nu(H) \leq 1$,
- (ii) $\delta(H_v) \geq 2$ with equality if $\nu(H) \leq 1$,
- (iii) $d^2(v) = 5 \Rightarrow N(C_4; H, v) = 0$, and
- (iv) $d^2(v) = 6 \Rightarrow N(C_4; H, N(v)) = 0$.

Proof. Let H and $v \in V(H)$ be as in the premises. By Property 4 we have that $\delta(H) \geq 2$ and therefore $4 \leq d^2(v)$. Note that H contains no 2-regular component by Property 7 and the assumption that $C_5 \notin \mathcal{C}(H)$.

Suppose that $d^2(v) = 4$. Because H contains no 2-regular component there is a vertex $v' \in V(H)$ such that $d(v') = 2$, $d^2(v') \geq 5$ with $w \in N(v')$ such that $d(w) = 2$ and $d^2(w) = 4$ (take v' to be an endpoint in the path-component of $H[\{v \in V(H) : d(v) = 2\}]$ which contains v). Let x be the vertex in $N(w) \setminus \{v'\}$, the situation is then as illustrated in Figure 4.

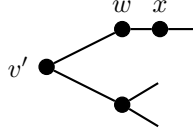


FIGURE 4. Local structure in the neighbourhood of v' .

But then we would have that

$$\nu(H_{v'}) \leq \nu(H) + 1 \leq 3,$$

whence $\delta(H_{v'}) \geq 2$ by Property 4 which contradicts that $d(H_{v'}; x) = 1$. Hence $d^2(v) \geq 5$. Moreover, if $\nu(H) \leq 1$ then $d^2(v) \leq 5$ as well since otherwise we would get $\nu(H_v) \leq \nu(H) - 2 - N(C_4; H, N(v)) \leq -1$, by Property 1, contradicting assumption (A1).

For the upper bound in (i) note that if $d^2(v) \geq 7$ then we would have $\nu(H_v) \leq \nu(H) - 5 \leq -3$ which contradicts assumption (A1).

The first assertion of (ii) follows by Property 4 since we have $\nu(H_v) \leq \nu(H) + 1 \leq 3$. The second assertion of (ii) since $\nu(H_v) \leq 1$ would contradict Property 4 because we have $\nu(H_v) \leq \nu(H) + 1 - N(C_4; H, N(v)) \leq 2$ by Property 1.

For (iii), suppose that $d^2(v) = 5$. If $N(C_4; H, v) \geq 1$ then $N(C_4; H, w) \geq 1$ where w is the bivalent neighbour of v . We must then have that $d^2(w) \geq 6$ by (ii). Hence, $\nu(H_w) \leq \nu(H) - 2 - 1 \leq -1$, which contradicts assumption (A1). Hence $N(C_4; H, v) = 0$.

Finally, if $d^2(v) = 6$ then $\nu(H_v) \leq \nu(H) - 2 - N(C_4; H, N(v)) \leq 0 - N(C_4; H, N(v))$ whence $N(C_4; H, N(v)) = 0$ or this would contradict assumption (A1). This proves (iv). \square

The following corollary is immediate.

Corollary 2. Let $H \leq G$ be a connected graph such that $\nu(H) \leq 2$. Suppose that there is a bivalent vertex $v \in V(H)$ with second valency four. Then $H \cong C_5$.

Property 14. Let $H \leq G$. If $\nu(H) \leq 2$ and $v \in V(H)$ is bivalent with second valency six, then $N_2(v)$ is a minimal destabiliser of H_v of size four.

Proof. Note that H is edge-critical by Property 2.

$N_2(v)$ clearly has size at most four and if it would have size less than four then there would be a cycle of length four through v , contradicting Property 13.

It follows from Lemma 3 that $N_2(v)$ is a destabiliser. If it were not a *minimal* destabiliser then, $N_2(v) \setminus \{x\}$ would destabilise H_v for some $x \in N_2(v)$. Let H' be the graph obtained when we remove the edge between x and $N(v)$ from H . By Lemma 1 we then have that $\alpha(H') = \alpha(H) + 1$ and if S is a maximum independent set in H' , then it must contain both w_1 and x .

Since $S \cap V(H'_v) = S \cap V(H_v)$ is an independent set we must have that $|S \cap V(H_v)| \leq \alpha(H_v) = \alpha(H) - 1$ and therefore S contains both w_1 and w_2 . But $|S \cap V(H_v)| \leq \alpha(H) - 1$ and $S \cap V(H_v) \cap (N_2(v) \setminus \{x\}) = \emptyset$, contradicting that $N_2(v) \setminus \{x\}$ is a destabiliser. \square

Property 15. If $H \leq G$ is a connected graph which contains two adjacent bivalent vertices, v_1 and v_2 , each of second valency five, then

- (i) $\nu(H) \leq 1 \Rightarrow H \cong Ch_k$ for some $k \geq 3$,
- (ii) $\nu(H) \leq 2 \Rightarrow N(C_5; H, v_1) = N(C_5; H, v_2) = 2$,

whence, in particular, there is at least one cycle of length four through the trivalent neighbours of v_1 and v_2 .

Proof. The assertion is trivially true for $H = K_1$. Suppose that it holds for all $J \leq G$ with fewer vertices than H . Moreover, let H be connected with $\nu(H) \leq 2$ and $\delta(H) = 2$ but suppose that either (i) or (ii) does not hold for H .

We then have, a fortiori, $H \not\cong C_5 \cong Ch_2$. Therefore, by Property 13, we have that $\delta(H_{v_1}), \delta(H_{v_2}) \geq 2$ and that there is no cycle of length four through v_1 or v_2 . Denote the trivalent neighbour of v_i by u_i and the two neighbours of u_i that are not v_i by w_{i1} and w_{i2} for $i \in \{1, 2\}$. We then have that

$$(1) \quad d(w_{ij}) \geq 3 \quad (\forall i, j \in \{1, 2\}).$$

The neighbourhood of v_1 and v_2 in this situation has been illustrated in Figure 5.

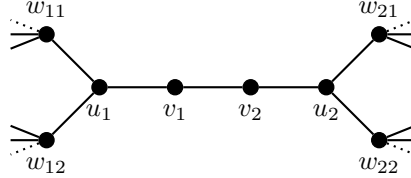


FIGURE 5. The neighbourhood of v_1 and v_2 in H .

Note however that the vertices w_{ij} are not, a priori, distinct. The vertices w_{i1} and w_{i2} are however distinct.

Note also that for any subgraph $J < H$ such that $\nu(J) \leq 1$ and $\delta(J) = 2$ every bivalent vertex in J must have second valency five by Property 13 or belong to a $Ch_2 \cong C_5$ -component of J . Hence, inductively, any bivalent vertex in J belongs to a Ch_k -component for some $k \geq 2$. This fact will be used repeatedly throughout this proof.

Case 1: $N(u_1) \cap N(u_2) = \emptyset$. Then all four vertices w_{ij} are distinct and there is no cycle of length five through the v_1 and v_2 . Suppose that w_{11} had a bivalent neighbour y , then $\nu(H_{u_1, v_2}) \leq \nu(H) + 9 - 10 \leq 1$ and $d(H_{u_1, v_2}; y) = 1$, contradicting Property 4. Hence,

$$(2) \quad w_{ij} \text{ has no bivalent neighbours} \quad (\forall i, j \in \{1, 2\}).$$

If $d(w_{ij}) \geq 4$ for some $i, j \in \{1, 2\}$ then without loss of generality assume that $d(w_{11}) \geq 4$. Hence $\nu(H_{v_2}) \leq \nu(H) + 1 \leq 3$, but $d^2(H_{v_2}; u_1) \geq 7$ which would give

$\nu(H_{v_2, u_2}) \leq \nu(H_{v_2}) - 5 \leq -2$, contradicting assumption (A1). Therefore $d(w_{ij}) = 3$ for all $i, j \in \{1, 2\}$.

If $d^2(w_{11}) \geq 10$ then we get $\nu(H_{w_{11}}) \leq \nu(H) + 3 \leq 5$ and thus v_2 would be monovalent in $H_{w_{11}}$ by $d(H_{w_{11}}; v_1) = 1$ and Property 9. It would then follow that w_{11} is adjacent to u_2 contradicting our assumption that $N(u_1) \cap N(u_2) = \emptyset$. Hence, by (2) and an analogous argument for $w_{ij} \neq w_{11}$, we get

$$d^2(w_{ij}) = 9 \quad (\forall i, j \in \{1, 2\})$$

This means that $\nu(H_{v_1, w_{11}}) \leq \nu(H) + 1 - 2 \leq 1$ and $d(H_{v_1, w_{11}}; u_2) = 2$. Note that $d^2(H_{v_1}; w_{11}) = 6$ since the w_{ij} are distinct. By Lemma 5 we get that H_{v_1} is connected. It is however possible that $H_{v_1, w_{11}}$ is not since H_{v_1} might not be edge-critical.

If $H_{v_1, w_{11}}$ is connected, then since u_2 is bivalent in $H_{v_1, w_{11}}$ we must have, by the inductive hypothesis, that $H_{v_1, w_{11}} \cong Ch_k$ for some $k \geq 2$. Also, $N_2(H_{v_1}; w_{11})$ destabilises $H_{v_1, w_{11}}$ and thus, by Lemma 9, either $N_2(H_{v_1}; w_{11})$ contains a pair of adjacent bivalent vertices of $H_{v_1, w_{11}}$ or $N(C_4; G, N(v)) \geq 2$. The latter is not possible however since then we would get $\nu(H_{v_1, w_{11}}) \leq \nu(H) + 1 - 2 - (2) \leq -1$, contradicting assumption (A1). Then since w_{12} and u_2 are bivalent in H_{v_1} , and not adjacent to w_{11} , we would have to have $w_{12}u_2 \in E(H)$, contradicting $N(u_1) \cap N(u_2) = \emptyset$.

On the other hand, if $H_{v_1, w_{11}}$ is disconnected, then every component is 2-stable by Property 6. Therefore $e(C, H \setminus C) \geq 3$ for all $C \in \mathcal{C}(H_{v_1, w_{11}})$, but $e(N[v_1, w_{11}], H \setminus N[v_1, w_{11}]) = 6$. Hence there are exactly two components, C_1 and C_2 , in $H_{v_1, w_{11}}$ each connected to the rest of the graph by three edges. Moreover $N_2(H_{v_1}; w_{21})$ destabilises one of the components, say C_1 , since otherwise $\alpha(H_{v_1}) \geq \alpha(H_{v_1, w_{11}}) + 2$ by Lemma 2 and then $\nu(H_{v_1, w_{11}}) \leq \nu(H_{v_1}) - 35 \leq -32$, contradicting assumption (A1).

By induction, Property 11 and Corollary 1 we get that both C_1 and C_2 are Ch_k -components. Since $e(N(H_{v_1}; w_{11}), V(C_1)) = 3$ we get by Lemma 7 that $N_2(H_{v_1}; w_{11}) \cap N(C_1) = N_{C_1}[x]$ for some bivalent vertex x in C_1 . Thus by the recursive construction of Ch_k -graphs we get that $H' := H[V(C_1) \cup N_{H_{v_1}}[w_{11}]] \cong Ch_{k+1}$ where $k \geq 2$ is such that $C_1 \cong Ch_k$. But $e(H', H \setminus H') = 2$, since $e(N(v_1), V(C_2)) = 2$, contradicting Corollary 1.

Case 2: $|N(u_1) \cap N(u_2)| = 1$. Without loss of generality assume that $w := w_{11} = w_{21}$. The situation is then as illustrated in Figure 6.

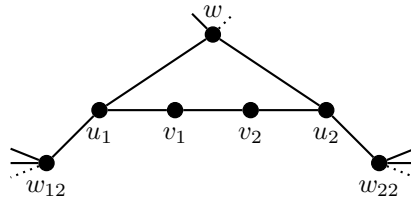
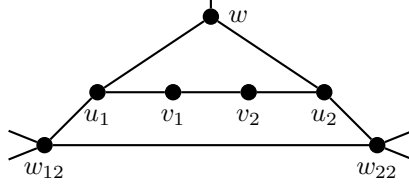


FIGURE 6. The neighbourhood of v_1 and v_2 in H .

If w had a bivalent neighbour, y , then since $\nu(H_{u_1, v_2}) \leq \nu(H) + 9 - 7 \leq 4$ we get that y has a monovalent neighbour y' in H_{u_1, v_2} by Property 9. But y' is not adjacent to any vertex in $N(u_1) \cup N(u_2) \setminus \{w_{12}\}$, whence $y'w_{12} \in E(H)$. Then we would have that $\nu(H_{u_2, v_1, y}) \leq \nu(H) + 9 - 7 - 7 \leq -3$ since y' has valency at least two in H_{u_2, v_1} . However, this contradicts assumption (A1), whence w has no bivalent neighbours.

FIGURE 7. The neighbourhood of v_1 and v_2 in H .

If w_{12} had a bivalent neighbour, y , then $\nu(H_{u_1, v_2}) \leq \nu(H) + 9 - 7 \leq 4$ and therefore y would have a monovalent neighbour, y' in H_{u_1, v_2} (by Property 9). But y' would then be adjacent to w and no other vertex in $N(u_1) \cup N(v_2)$, contradicting that w has no bivalent neighbour. By an completely analogous argument for w_{22} we can now say that

(3) w_{ij} has no bivalent neighbours. $(\forall i, j \in \{1, 2\})$

Subcase 2.1: $w_{12}w_{22} \in E(H)$. If $d(w_{12}) = 3$ then $d^2(w_{12}) \geq 9$ by (3) and therefore $\nu(H_{w_{12}, v_1}) \leq \nu(H) + 6 - 7 \leq 1$ but u_2 is monovalent in H_{w_{12}, v_1} , contradicting Property 4. Hence, $d(w_{12}) \geq 4$ and analogously we get that $d(w_{22}) \geq 4$. If $d(w_{12}) \geq 5$ then we would get $\nu(H_{u_1, v_2}) \leq \nu(H) + 3 - 7 \leq -3$, contradicting the assumption (A1). By an analogous argument again for w_{22} we can now conclude that $d(w_{12}) = d(w_{22}) = 4$, i.e. we have the situation as illustrated in Figure 7.

If $d(w) \geq 4$ then $\nu(H_{u_1, v_2}) \leq \nu(H) + 3 - 7 \leq -2$, contradicting assumption (A1). Hence, $d(w) = 3$. If $N(w_{12}) \cap N(w) = \{u_1\}$ then $\nu(H_{w_{12}, v_1, u_2}) \leq \nu(H) + 11 - 7 - 7 \leq -1$, contradicting assumption (A1). Thus, by the previous and an analogous argument for w_{22} we must have that

$$N(w_{12}) \cap N(w) \neq \{u_1\} \text{ and } N(w_{22}) \cap N(w) \neq \{u_2\}.$$

But since $d(w) = 3$ we then must have that $N(w_{22}) \cap N(w) \setminus \{u_2\} = N(w_{12}) \cap N(w) \setminus \{u_1\}$. Let a be an element of $N(w_{22}) \cap N(w) \setminus \{u_2\}$, then we would have a triangle through w_{12}, a and w_{22} , contradicting the assumption that G is triangle-free.

Subcase 2.2: $w_{12}w_{22} \notin E(H)$. Since $\nu(H_{v_i}) \leq \nu(H) + 1 \leq 3$ we must have that $d^2(H_{v_i}; u_{3-i}) \leq 7$ for $i \in \{1, 2\}$. Otherwise we would get $\nu(H_{v_i, u_{3-i}}) \leq \nu(H) + 1 - 5 \leq -2$, contradicting assumption (A1).

Suppose that $d(w) \geq 4$, then $d(H_{v_i}; w_{3-i, 2}) \leq 3$ for $i \in \{1, 2\}$. But then $d(H_{v_i}; w_{3-i, 2}) = 3$ by (1) and $d(H_{v_i}; w_{3-i, 2}) = d(w_{3-i, 2})$. Now note that $\nu(H_{v_2, u_1}) \leq \nu(H) + 1 - 2 \leq 1$ and w_{22} is bivalent in H_{v_2, u_1} . Hence $w_{22} \in V(C')$ for some $C' \in \mathcal{C}(H_{v_2, u_1})$ such that $C' \cong Ch_k$, where k is at least two. Let $N := N[u_1, v_2]$. If H_{v_2, u_1} contained more than one component, say some $C' \neq C$, $C' \in \mathcal{C}(H_{v_2, u_1})$, then C' would have at most two edges to N in H . But C' has to be 2-stable by Property 6, so we would have redundant edges in $E(N, H \setminus N)$ by Corollary 1.

Hence $H_{v_2, u_1} \cong Ch_k$ and $N_2(H_{v_2}; u_1)$ destabilises H_{v_2, u_1} , by Property 12. Let $T := N_2(H_{v_2}; u_1)$. If $|T| = 4$ and T is a minimal destabiliser, then either T is connected in H_{v_2, u_1} or $k = 3$ and $T = V_2(H_{v_2, u_1})$ by Lemma 8. The latter is however not possible since w_{22} is bivalent in H_{v_2} . The former is not possible either since then T would induce one of the three connected triangle-free graphs of size four (P_4 , $K_{1,3}$ or C_4) in H_{v_2, u_1} . It is easy to see that in all three cases $N(C_4; H_{v_2}, N(u_1)) \geq 2$ and thus $\nu(H_{v_2, u_1}) \leq \nu(H) + 1 - 2 - (2) \leq -1$, contradicting assumption (A1). Hence we get by Lemmas 7 and 8 that $N_C[x] \subseteq T$ for some $x \in V_2(H_{v_2, u_1})$. Therefore by the recursive construction of Ch_k we must have that $H_{v_2} \cong Ch_{k+1} + \{\alpha, \beta\}$ for some $\{\alpha, \beta\} \in \binom{V(Ch_{k+1})}{2} \setminus E(Ch_{k+1})$. The extra edge $\{\alpha, \beta\}$ is incident to

either w or w_{12} . In either case $\{\alpha, \beta\}$ is incident to at least one bivalent vertex in Ch_{k+1} , since both w and w_{12} are trivalent in H_{v_2} . But then $\{\alpha, \beta\}$ is also incident to the bivalent vertex in $V_2(Ch_{k+1}) \setminus \{u_2, w, w_{12}, w_{22}\}$, since otherwise that vertex would be bivalent also in H , but have second valency 6, and a cycle of length four through its neighbourhood. Hence $\{\alpha, \beta\} \subseteq V_2(Ch_{k+1})$. Now, note that $S := \{u_1, w, w_{22}\} \subseteq V(H_{v_2})$ is such that $|S| = 3$ and $|S \cap V_2(Ch_{k+1})| \geq 2$ since both u_1 and w_{22} are bivalent in H_{v_2} (and therefore also in $H_{v_2, w_{11}}$). Furthermore, if $|S \cap V_2(Ch_{k+1})| = 2$ then $S \cap V_2(Ch_{k+1}) = \{u_1, w_{22}\}$ and u_1 is not adjacent to w_{22} , by assumption. By Property 12 therefore H_{v_2} is not destabilised by $S = \{u_1, w, w_{22}\} = N_2(v_2)$, contradicting Lemma 3.

Hence $d(w) = 3$. Let y denote the neighbour of w that is neither u_1 nor u_2 . This situation is illustrated in Figure 8.

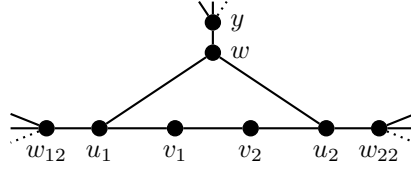


FIGURE 8. The neighbourhood of v_1 and v_2 in H in Subcase 2.2.

Subcase 2.2.1: $d(w_{12}) = 3$ and $d(w_{22}) \geq 4$ (or analogously $d(w_{12}) \geq 4$ and $d(w_{22}) = 3$). We have $\nu(H_{v_1, u_2}) \leq \nu(H) + 1 - 2 \leq 1$ and thus $d(w_{22}) = 4$ (otherwise $\nu(H_{v_1, u_2}) \leq -2$, contradicting assumption (A1)). Inductively w is a vertex of some $C \in \mathcal{C}(H_{v_1, u_2})$ such that $C \cong Ch_k$ for some $k \geq 2$. Moreover, H_{v_1, u_2} is connected since every component of H_{v_1, u_2} is 2-stable and otherwise $e(C, H \setminus C) \leq 2$, contradicting Corollary 1.

Since w_{22} has three neighbours in C we must have that $k \geq 3$, since H is triangle-free. Note that $N_2(H_{v_1}; u_2)$ destabilises C , but $|N_2(H_{v_1}; u_2)| \leq 4$ and is not adjacent to the, in H_{v_1, u_2} , bivalent w_{12} . Now, let $T := N_2(H_{v_1}; u_2)$. If $|T| = 4$ then either T is not minimal or T is connected in H_{v_1, u_2} , by Lemma 8. But in the latter case $H_{v_1, u_2}[T] \in \{P_4, K_{1,3}, C_4\}$ so $N(C_4; H_{v_1}, N(u_2)) \geq 2$. However, then $\nu(H_{v_1, u_2}) \leq \nu(H) + 1 - 2 - (2) \leq -1$, contradicting assumption (A1).

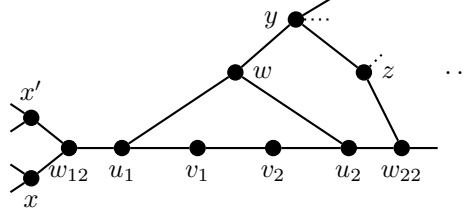
Hence $N_2(H_{v_1}) \supseteq N_C[x]$, $x \in V_2(C)$ and also $N_2(H_{v_1}; u_2)$ contains a, in C , bivalent neighbour of w_{12} , by Lemmas 7 and 8. Therefore y is trivalent in H with at least two cycles of length four through its neighbourhood, and second valency at least 10, which would give

$$\nu(H_{y, w_{22}, v_2}) \leq \nu(H) + 3 - (2) + 1 - 7 \leq -3,$$

contradicting assumption (A1).

Subcase 2.2.2: $d(w_{12}) = d(w_{22}) = 3$. By (3) we have that $d(w_{22}) \geq 9$ and if $d^2(w_{22}) \geq 10$ we would get $\nu(H_{w_{22}, v_2}) \leq \nu(H) + 3 - 7 \leq -2$, contradicting assumption (A1). Hence $d^2(w_{22}) = 9$ and $\nu(H_{w_{22}, v_2}) \leq 1$. Similarly, we also have that $\nu(H_{w_{12}, v_1}) \leq 1$. Since w and u_2 are bivalent neighbours in H_{w_{12}, v_1} they, inductively, belong to some Ch_k -component, $C \in \mathcal{C}(H_{w_{12}, v_1})$ for $k \geq 2$ and also w, u_1 are adjacent bivalent vertices in a component C' of H_{w_{22}, v_2} such that $C' \cong Ch_\ell$ for some $\ell \geq 2$. Let $\{z\} = N_C(w_{22}) \setminus \{u_2\}$. Note that $\{y\} = N_C(w) \setminus \{u_2\} = N_{C'}(w) \setminus \{u_w\}$. Furthermore, let $\{x, x'\} := N(w_{12}) \setminus \{u_1\}$. This situation is illustrated in Figure 9.

Suppose $k = 2$. By (3) both y and z are at least trivalent in H . Therefore y, z and w_{22} are all adjacent to either x or x' . Since H is triangle-free this means that

FIGURE 9. The neighbourhood of v_1 and v_2 in H in subcase 2.2.2.

y and w_{22} are adjacent to the same neighbour of w_{12} , say x , and z is adjacent to x' . But in this case x, w_{22}, z and y forms a cycle of length four and we get $\nu(H_{w_{22},y}) \leq \nu(H) + 6 - (1) - 7 \leq 0$. However, v_2 is monovalent in $H_{w_{22},y}$, contradicting Property 4.

Hence $k \geq 3$ and by an analogous argument to the one for $k = 2$ we may assume that $\ell \geq 3$. In particular w_{12} and w_{22} both have two common neighbours with y , since they do in C' and C , respectively. These vertices are clearly distinct. There are also at least two cycles of length four through y , whence $\nu(H_{w,v_1}) \leq \nu(H) + 0 - (2) - 4 \leq -4$, contradicting assumption (A1).

Subcase 2.2.3: $d(w_{12}), d(w_{22}) \geq 4$. Then $d(w_{12}) = d(w_{22}) = 4$ since otherwise $\nu(H_{u_i, v_{3-i}}) \leq \nu(H) + 3 - 7 \leq -3$ for some $i \in \{1, 2\}$, which would contradict assumption (A1). We then have $\nu(H_{u_i, v_{3-i}}) \leq \nu(H) + 6 - 7 \leq 1$ for both $i \in \{1, 2\}$. If u had a bivalent neighbour, z , then z does not have a common neighbour with u_1 or u_2 (since if it did, then it would have second valency at least seven). Thus $\nu(H_{z,w,v_1}) \leq \nu(H) + 1 - 2 - 4 \leq -4$, contradicting assumption (A1). Hence y has no bivalent neighbour in H .

Let $H' := H[N[v_1, u_2]]$, then $e(H', H \setminus H') = 5$ and therefore H_{u_2, v_1} has at most one component. Otherwise there would be some component C' of H_{u_2, v_1} such that $e(C', H \setminus C') \leq 2$, contradicting Corollary 1 since H is edge-critical and C' is 2-stable by Property 6. Analogously, H_{v_2, u_1} is connected.

Suppose that $d(y) = 3$. Then $\nu(H_{u_2, v_1}) \leq 1$ and $d(H_{u_2, v_1}; y) = 2$. Hence y , inductively, belongs to some Ch_k -component, C , of H_{u_2, v_1} for some $k \geq 2$. By the argument above $C = H_{u_2, v_1}$. Since $d(H_{u_2, v_1}; w_{12}) = 3$ we obtain that $N_2(v_1) \cup N_2(u_2)$ does not contain all bivalent vertices of C (because then two adjacent ones would have to be in $N(w_{22})$, contradicting that H is triangle-free). Clearly $k \geq 3$ since w_{22} has three neighbours in C . Also, $T := N_2(H_{v_1}; u_2)$ destabilises C by Property 12. Now, $|T| \leq 4$ and if $|T| = 4$ with T connected in H_{v_1, u_2} then $H_{v_1, u_2}[T]$ must be $P_3, K_{1,3}$ or C_4 and therefore $N(C_4; H_{v_1}, N(u_2)) \geq 2$. However, then $\nu(H_{v_1, u_2}) \leq \nu(H) + 1 - 2 - (2) \leq -1$, contradicting the assumption (A1). Therefore we get by Lemmas 7 and 8 that $N_C[x] \subseteq N_2(H_{v_1}; u_2)$ for some, in C , bivalent vertex x .

Note that w_{12} is not adjacent to y since if $w_{12}y \in E(H)$ then $\nu(H_{v_2, u_1}) \leq \nu(H) + 1 - 2 - (1) \leq 0$ but y would be at most monovalent in H_{v_2, u_1} , contradicting Property 4. Analogously we can show that $w_{22}u \notin E(H)$. Since y has no bivalent neighbour in H but has a bivalent neighbour in H_{v_1, u_2} we must have $x = y$. Hence w_{22} has two common neighbours with y . Analogously one gets that w_{12} has two common neighbours with y also in C since w_{12} is not adjacent to w . This is not possible since there no cycles of length four through bivalent vertices in $C \cong Ch_k$.

Hence $d(y) \geq 4$. If $yw_{12}, yw_{22} \in E(H)$ then $\nu(H_{v_2, u_1}) \leq \nu(H) + 1 - (1) - 2 - (1) \leq -1$, contradicting assumption (A1). Hence at most one of w_{12} and w_{22} is adjacent to y . Suppose that $yw_{12} \in E(H)$, then $yw_{22} \notin E(H)$ and therefore

$\nu(H_{v_2, u_1}) \leq +1 - (1) - 2 \leq 0$. Moreover $d(H_{v_2, u_1}; y) = 2$, $d(H_{v_2, u_2}; w_{22}) = 3$ and H_{v_2, u_1} is connected. Hence, inductively, $H_{v_2, u_1} \cong Ch_k$ for some $k \geq 3$. In particular, y has a bivalent neighbour z in H_{v_2, u_1} . If $d(z) \geq 3$ then z would have to be adjacent to w_{12} , but that would give a triangle through w_{12}, z and y . Hence $d(z) = 2$ and $d^2(z) \geq 4 + 3$, since z has a trivalent neighbour in H_{u_1, v_2} because $k \geq 3$. Thus $yw_{12} \notin E(H)$ and analogously $yw_{22} \notin E(H)$.

Subcase 2.2.3.1: y has a trivalent neighbour except for w . Let z be such a trivalent neighbour. Since $d^2(w) = 10$ we get that $\nu(H_{w, v_1}) \leq \nu(H) + 3 - 4 \leq 1$ and z is bivalent in H_{w, v_1} . So, inductively, z belongs to some Ch_k -component, C , for $k \geq 2$ of H_{w, v_1} . Suppose that $C \cong Ch_2$. If both neighbours, x_1 and x_2 , of z in C were bivalent in H , then we would have that $\delta(H_{x_i}) = 1$, contradicting Property 13. Hence, z has a neighbour which has lower valency in C than in H . If $C \cong Ch_k$ for some $k \geq 3$ then the bivalent neighbour of z in C is not bivalent in H because then it would have second valency at least six and a cycle of length four through its neighbourhood. Hence, also in this case we have that z has a neighbour which has lower valency in C than in H . Let us call such a neighbour z' . Since no neighbour of z can be adjacent to y (or we would get a triangle in H) we must have that z' is non-adjacent to y and therefore adjacent to u_1 or u_2 . Hence, $z' = w_{12}$ or $z' = w_{22}$, in either case we get that $d(H; z') = 4$ but then $d(H_{w, v_1}; z') = 3$, contradicting that z' is bivalent in H_{w, v_1} .

Subcase 2.2.3.2: w is the only trivalent neighbour of y . In this case we have that $\nu(H_y) \leq \nu(H) + 5 \leq 7$. If $d(H_y; w_{i2}) \geq 4$, then $\nu(H_{y, u_i, v_{3-i}}) \leq \nu(H) + 5 - 2 - 7 \leq -2$, contradicting the assumption (A1). Hence, $d(H_y; w_{i2}) \leq 3$ for $i \in \{1, 2\}$. If $d(H_y; w_{i2}) \leq 2$ for some $i \in \{1, 2\}$, then y and w_{i2} would both have two common neighbours. If both w_{12} and w_{22} have two common neighbours with y , then $\nu(H_y) \leq \nu(H) + 5 - (2) \leq 5$. All vertices $w_{12}, u_1, v_1, v_2, u_2$ and w_{22} would be bivalent in H_y . Now, since $w_{12}w_{22}$ we must have that the component, C , containing w_{12} in H_y is not 2-regular by Property 7. Hence C contains some bivalent vertex, x , that is an endpoint of a path of bivalent vertices in C of length at least five. Say z is a vertex at distance 2 from x in the path of bivalent vertices, then $\nu(H_{y, x, z}) \leq \nu(H) + 5 - (2) + 1 - 7 \leq -1$, contradicting assumption (A1).

Hence not both w_{12} and w_{22} have two common neighbours with y . Suppose that w_{12} has two common neighbours with y . Then $\nu(H_{y, u_2, v_1}) \leq \nu(H) + 5 - (1) + 1 - 7 \leq 0$ but w_{12} would be at most monovalent in H_{y, u_2, v_1} , contradicting Property 4. Therefore w_{12} has exactly one common neighbour with y . Analogously w_{22} also has one common neighbour with y .

Suppose β_i is the common neighbour of w_{i2} and y for $i \in \{1, 2\}$ and let β_3 be remaining neighbour of y that is not w , as has been illustrated in Figure 10.

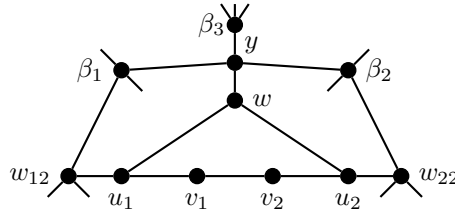


FIGURE 10. The neighbourhood of v_1 and v_2 in H in subcase 2.2.3.2.

We then have that $\nu(H_{y, u_i, v_{3-i}}) \leq \nu(H) + 5 + 1 - 7 \leq 1$ and $w_{(3-i), 2}$ is bivalent in $H_{y, u_i, v_{3-i}}$ for $i \in \{1, 2\}$. Therefore, inductively w_{22} belongs to a Ch_k -component, C , of H_{y, u_1, v_2} and w_{12} belongs to a Ch_l -component of H_{y, u_2, v_1} for some $k, l \geq 2$.

Suppose $k = 2$. Let a_1 and a_4 be the neighbours of w_{22} in C . Also let a_2 and a_3 be the neighbours of a_1 and a_4 in C that are not w_{22} , respectively. The situation is then like what is illustrated in Figure 11.

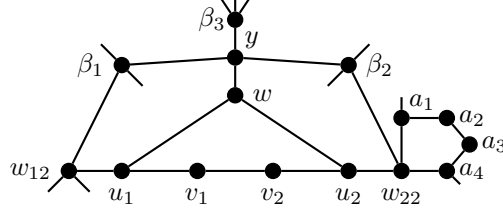


FIGURE 11. The neighbourhood of v_1 and v_2 in H if $k = 2$ in subcase 2.2.3.2.

Now, since w_{22} has no bivalent neighbours both a_1 and a_4 have valency at least three in H . Hence, a_1 and a_4 have neighbours in $\{\beta_1, \beta_2, \beta_3, w_{12}\}$.

If $d(a_1) = 3$ and $d(a_2) = 2$ then $\nu(H_{a_1}) \leq \nu(H) + 3 \leq 5$ and a_4 would have to be monovalent, by Property 9, in H_{a_1} as its neighbour a_3 is monovalent. Then a_1 and a_4 must have a common neighbour in $\{\beta_1, \beta_2, \beta_3, w_{12}\}$. But then $\nu(H_{a_1, a_3}) \leq \nu(H) + 3 - (1) - 4 \leq 0$ while $d(H_{a_1, a_3}; u_2) = 2$, $d^2(H_{a_1, a_3}; u_2) = 5$, $d(H_{a_1, a_3}; v_2) = 2$ and $d^2(H_{a_1, a_3}; v_2) = 4$, which is impossible since u_2 and v_2 should inductively belong to the same $Ch_{k'}$ -component of H_{a_1, a_3} .

If, on the other hand, $d(a_1) = 3$ and $d(a_2) \geq 3$, then $\nu(H_{a_1}) \leq \nu(H) + 0 \leq 2$. If $d(a_2) \geq 4$ then a_1 would have second valency twelve in H , giving $\nu(H_{a_1}) \leq \nu(H) - 3 \leq -1$, contradicting assumption (A1). Hence $d(a_2) = 3$. Clearly $d(H_{a_1}; a_3) \geq 2$ and therefore $d(a_3) \geq 3$. If $d(a_3) \geq 4$ then $\nu(H_{a_2}) \leq \nu(H) - 3 \leq -1$, contradicting assumption (A1). Hence $d(a_2) = d(a_3) = 3$.

Now, u_2 is bivalent in H_{a_2} and H_{a_3} of second valency five but v_2 is bivalent of second valency four. Hence u_2 does not belong to a Ch_k -component in H_{a_2} or H_{a_3} . In particular, $\nu(H_{a_1}), \nu(H_{a_4}) \geq 2$ since otherwise this would contradict the inductive assumption by Property 13. But if either a_3 or a_2 were adjacent to β_2 there would be a cycle of length four through that vertex. On the other hand if neither a_2 nor a_3 is adjacent to β_2 then all four vertices (since H is triangle-free) a_1, a_2, a_3 and a_4 are adjacent to vertices in $\{w_{12}, \beta_1, \beta_3\}$. It is easy to see that we then must get a cycle of length four through a_1 or a_4 . Thus, independently of whether either a_2 or a_3 is adjacent to β_2 , we get $\nu(H_{a_1}) \leq \nu(H) + 0 - (1) \leq 1$ or $\nu(H_{a_4}) \leq \nu(H) + 0 - (1) \leq 1$, in either case a contradiction.

Hence $d(a_1) \geq 4$ and analogously we get that $d(a_4) \geq 4$. So $\nu(H_{w_{22}, v_1}) \leq \nu(H) + 5 - 7 \leq 0$. Thus neither a_2 nor a_3 is bivalent in H (or they would be monovalent in H_{w_{22}, v_1}). Therefore $e(V(C), \{\beta_1, \beta_2, \beta_3, w_{12}\}) \geq 5$, whence some vertex in $\{\beta_1, \beta_2, \beta_3, w_{12}\}$ is adjacent to two vertices in $V(C)$. These two vertices must form an independent set in C and therefore are at distance two in C . Hence $N(C_4; H, N(w_{22})) \geq 1$ which gives us $\nu(H_{w_{22}, v_1}) \leq \nu(H) + 5 - (1) - 7 \leq -1$, contradicting the assumption (A1).

Hence, $k \geq 3$ and the graph looks like in Figure 12.

Let a_1 be the bivalent neighbour of w_{22} in C , while a_4 is the trivalent neighbour of w_{22} in C . a_1 is at least trivalent in H and therefore adjacent to one of the vertices in $\{\beta_1, \beta_2, \beta_3, w_{12}\}$, which are all tetravalent. Then if $d(a_1) = 3$ we would have $d^2(a_1) \geq 3 + 4 + 4 = 11$, whence $\nu(H_{a_1}) \leq \nu(H) + 0 - (1) \leq 1$, since there is a cycle of length four through the trivalent neighbour of a_1 in C . However, v_2 would be bivalent, in H_{a_1} , with second valency four but not belonging to a

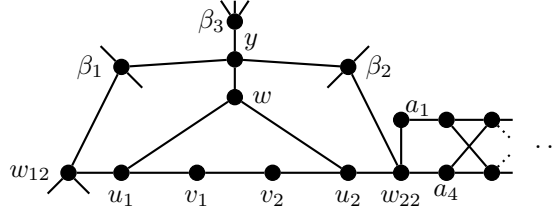


FIGURE 12. The neighbourhood of v_1 and v_2 in H if $k \geq 3$ in subcase 2.2.3.2.

C_5 -component since w is trivalent, contradicting Property 13. Hence $d(a_1) \geq 4$. Suppose that $d(a_4) = 3$. There is a cycle of length four through a_4 and we have $d^2(a_4) \geq 4 + 3 + 3 = 10$. We also have that $a_4 w_{12} \notin E(H)$ since a_4 also is trivalent in C . So in particular v_1 has second valency two in H_{a_4, u_2} and thus we have $\nu(H_{a_4, u_2, v_1}) \leq \nu(H) + 3 - (1) + 1 - 7 \leq -3$, contradicting the assumption (A1). Hence also $d(a_4) \geq 4$. So also in the case when $k \geq 3$ we get that the two neighbours of w_{22} in C have valency at least four in H . But then we have $d^2(w_{22}) \geq 15$ and therefore $\nu(H_{w_{22}, v_2}) \leq \nu(H) + 5 - (1) - 7 \leq -1$, contradicting assumption (A1).

Case 3: $|N(u_1) \cap N(u_2)| = 2$. If $\nu(H) = 2$ we are done because we have two cycles of length five through v_1 and v_2 . Suppose therefore that $\nu(H) \leq 1$. Let w and w' denote the two common neighbours of u_1 and u_2 . Since they form, together with u_1 and u_2 , a cycle of length four we get by Property 13 that neither w nor w' have valency two. We then have the situation which is illustrated in Figure 13.

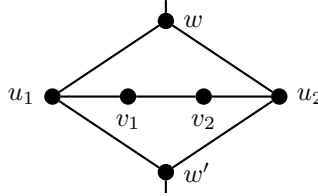


FIGURE 13. The neighbourhood of v_1 and v_2 in H in case 3.

Now, since $\nu(H_{v_1}) \leq \nu(H) + 1 - (1) \leq 1$, and u_2 is bivalent in H_{v_1} , we inductively get that u_2 belongs to some Ch_l -component of H_{v_1} , for some $l \geq 2$. But then $H \cong Ch_{l+1}$ by the recursive Definition 1, here u_2 corresponds to “ x ” in the definition and v_1, v_2 and u_1 correspond “ v ,” “ w_1 ” and “ w_2 ”, respectively. However, this contradicts the assumption that $H \not\cong Ch_k$ for all $k \geq 2$. \square

As a simple corollary of Property 15 we have the following property.

Property 16. Let $H \leq G$ be connected with $\delta(H) = 2$. If $\nu(H) \leq 1$ then $H \cong Ch_k$ for some $k \geq 2$.

Proof. By Property 13 either $H \cong C_5 = Ch_2$ or there is a vertex $v_1 \in V(H)$ such that $d^2(v) = 5$. Hence, by the first part of Property 15 we get that $H \cong Ch_k$ for some $k \geq 3$. \square

Property 17. If $\nu(G) \leq -1$ or $\nu(G) = 0$ but $G \notin \mathcal{G}$, then $\delta(G) \geq 3$.

Proof. Suppose that G satisfies the premises. By Property 4 we have $\delta(G) \geq 2$ and by Lemma 14 G is connected. Suppose that $\delta(G) = 2$, then by Property 16 we have

that $G \cong Ch_k$ for some $k \geq 2$, contradicting that $\nu(Ch_k) = 0$ for all $k \geq 2$. Hence, $\delta(G) \geq 3$. \square

We now show that in subgraphs $H \leq G$, with $\nu(H) \leq 2$, bivalent vertices with bivalent neighbours appear in “balanced pairs”, i.e. any two adjacent bivalent vertices have the same second valency.

Property 18. Let $H \leq G$ be such that $\nu(H) \leq 2$ and $v \in V(H)$ be bivalent with a bivalent neighbour $u \in N(v)$ then $d^2(v) = d^2(u)$.

Proof. The assertion is trivially true if v belongs to a C_5 -component of H . Therefore we may assume that $v \in V(H)$ has exactly one bivalent neighbour. By Property 13(i) we then have that $d^2(v), d^2(u) \in \{5, 6\}$. For symmetry reasons it is enough to show that if $d^2(v) = 6$ then $d^2(u) \geq 6$.

Let w be the tetravalent neighbour of v and x the neighbour of u that is not v . Suppose moreover, for a contradiction, that x is trivalent. Then x is bivalent in H_v and, since $\nu(H_v) \leq \nu(H) - 2 \leq 0$, $x \in C \cong Ch_k \in \mathcal{C}(H_v)$ for some $k \geq 2$ (by assumption (A1)).

By Property 14 $N_2(v)$ is a minimal destabiliser of size four of C . N_2 is not connected in H_v since otherwise it would induce a P_4 , $K_{1,3}$ or C_4 . In either of these three cases we would get $N(C_4; H, N(v)) \geq 2$ and then $\nu(H_v) \leq \nu(H) - 2 - (2) \leq -2$, contradicting assumption (A1). Thus by Lemma 8 we must have that $k = 3$ and $N_2(v) = V_2(C)$. But then w is adjacent to two adjacent vertices in C , contradicting that G is triangle-free. \square

Property 19. If $H \leq G$, $C_5 \notin \mathcal{C}(H)$, $\nu(H) \leq 2$ and there are no cycles of length four through any vertices of valency less than four in H , then every bivalent vertex of H has second valency six.

Proof. Since $C_5 \notin \mathcal{C}(H)$ there are no bivalent vertices of second valency four by Corollary 2. Assume that $v \in V(H)$ is bivalent with second valency five. Then the bivalent neighbour of v would have second valency five by Property 18. Therefore, by Property 15, there is some cycle of length four through the trivalent neighbour of v , contradicting the assumption. \square

We will later, under the extra assumption that $\nu(G) \leq -1$, show that G does not contain any cycles of length four. In that situation the above property gives us that every bivalent vertex in $H \leq G$ has second valency four or six.

Property 20. If $H \leq G$, $C_5 \notin \mathcal{C}(H)$, $\nu(H) \leq 2$ and there are no cycles of length four through vertices of valency less than four in H , then for any pair u, w of distinct bivalent vertices of H ; $\text{dist}(u, w) \in \{1, 3\}$.

Proof. Both u and v have second valency six by Property 19. If $\text{dist}(u, w) \geq 4$ then the removal of u and its neighbours from H_v does not affect the valency or second valency of w , whence $\nu(G_{v,u,w}) \leq \nu(H) + 3 - 2 - 2 \leq -2$, contradicting assumption (A1).

Hence $\text{dist}(u, w) \leq 3$. If $\text{dist}(u, w) = 2$ then $\nu(H_{v,u}) \leq \nu(G) + 3 - 2 \leq 0$ but w would be at most monovalent in $G_{v,u}$, contradicting Property 4. \square

Property 21. If $H \leq G$, $C_5 \notin \mathcal{C}(H)$, $\nu(H) \leq 2$, there are no cycles of length four through vertices of valency less than four in H , and $u \in V(H)$ is a bivalent vertex with a bivalent neighbour, then u and the bivalent neighbour of u are the only two bivalent vertices of H .

Proof. Suppose for a contradiction that u_1 and u_2 are adjacent bivalent vertices in H and that there is a third bivalent vertex x in H . Then since $\text{dist}(x, u_1), \text{dist}(x, u_2) \in$

$\{1, 3\}$ (by Property 20) we must have that $\text{dist}(x, u_1) = \text{dist}(x, u_2) = 3$ because H is triangle-free.

By Property 19 we must also have $d^2(H; x) = 6$ and therefore $\nu(H_x) \leq \nu(H) - 2 \leq 0$. Since u_1 and u_2 are adjacent bivalent vertices in H_x they belong to some Ch_k -component, C , of H_x for some $k \geq 2$. The local structure in the graph then looks like in Figure 14.

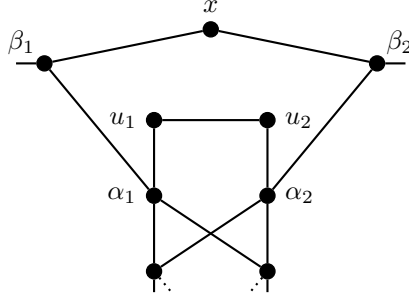


FIGURE 14. The neighbourhood of x in H .

Let α_1 and α_2 be the tetravalent neighbours of u_1 and u_2 in H , respectively. Then α_1 and α_2 are trivalent in H_x since otherwise x would have to have two common neighbours with α_i (for some $i \in \{1, 2\}$) in H and we would get $\nu(H_x) \leq \nu(H) - 2 - (1) \leq -1$, contradicting assumption (A1). Hence $k \geq 3$. Thus there is a cycle of length four through α_1 and $N(C_4; H, N(u_1)) > 0$ which gives $\nu(H_{u_1}) \leq \nu(H) + 3 - 2 - (1) \leq -1$, contradicting assumption (A1). \square

We now define another class of graphs, which will turn out to be useful. These graphs have ν -value two. It will later be shown that this class consists of all graphs with ν -value two among the connected triangle-free graphs with minimum valency two.

Definition 2. Let a_1, a_2, b_1 and b_2 denote the four bivalent vertices of $G = Ch_3$, where $a_1a_2, b_1b_2 \in E(G)$. We define the *shackled chain* SCh_1 by letting $V(SCh_1) = V(G) \cup \{v, w_1, w_2\}$ and $E(SCh_1) = E(G) \cup \{vw_1, vw_2, w_1a_1, w_1b_1, w_2a_2, w_2b_2\}$. For $k \geq 2$ we define SCh_k recursively as follows. Let a be any bivalent vertex in SCh_{k-1} with neighbours b_1 and b_2 . We then set $V(SCh_k) = V(SCh_{k-1}) \cup \{v, w_1, w_2\}$ and $E(SCh_k) = E(SCh_{k-1}) \cup \{vw_1, vw_2, w_1a, w_2b_1, w_2b_2\}$.

For example in Figure 15 the graphs SCh_1 and SCh_2 are shown.

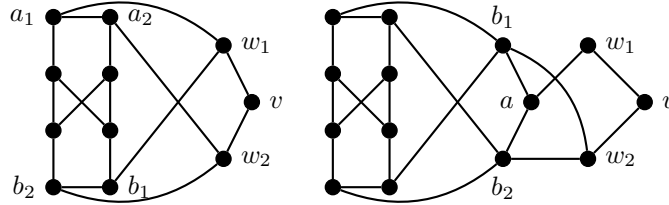


FIGURE 15. The smallest shackled chains SCh_1 (left) and SCh_2 (right).

It is not difficult to prove that the definition does not depend (up to isomorphism) on the choice of bivalent vertex in the recursive construction. It is also not hard to

see that $n(SCh_k) = 3k+8$, $e(SCh_k) = 5k+11$, $\alpha(SCh_k) = k+3$ and $N(C_4; SCh_k) = k$ for all $k \geq 1$. Therefore we get that $\nu(SCh_k) = 2$ for all $k \geq 1$.

We begin by showing that the chains and the shackled chains are the only connected graphs with ν -value no more than two and bivalent vertices.

Property 22. If $H \leq G$ is connected, $\nu(H) \leq 2$ and $\delta(H) = 2$, then $H \cong Ch_k$ for some $k \geq 2$ or $H \cong SCh_\ell$ for some $\ell \geq 1$.

Proof. The assertion is trivial for small graphs, i.e. for $H \leq G$ such that $n(H) = 1$. Assume that for all $J \leq G$ such that $n(J) < n(H)$ the assertion holds. Fix some bivalent vertex $v \in V(H)$. By Properties 4 and 13 we have $d^2(v) \in \{4, 5, 6\}$.

If $d^2(v) = 4$, then we are done by Corollary 2. If $d^2(v) = 6$ then $\nu(H_v) \leq \nu(H) - 2 \leq 0$ and $N(C_4; N(v)) = 0$, since otherwise $\nu(H_v) \leq -1$. We then have $|N_2(v)| = 4$ vertices at distance two from v . If $N_2(v)$ were connected, then it would induce one of the graphs C_4 , $K_{1,3}$ or P_4 , giving $N(C_4; N(v)) \geq 1$ in all cases. Hence $N_2(v)$ must be a disconnected minimal destabiliser (by Property 11 and since a non-minimal disconnected destabiliser of Ch_k ($k \geq 2$) would contain a bivalent vertex and its two neighbours, by Lemma 7).

It is easily checked that neither W_5 nor $(2C_7)_{2i}$ have destabilisers of size four that are disconnected. Also BC_k ($k \geq 5$) has only connected minimal destabilisers of size four (see e.g. [1, Lemma 6.3(e)]). The only possibility is that $\delta(H_v) = 2$ and $H_v \cong Ch_k$ for some $k \geq 2$. But then, by Lemma 8, $k = 3$ and $N_2(v) = V_2(Ch_k)$, which means precisely that $H \cong SCh_1$.

Finally, if $d^2(v) = 5$ then $N(C_4; N(v)) \geq 1$ by Property 15. Hence, $\nu(H_v) \leq \nu(H) + 1 - (1) \leq 2$. The bivalent neighbour of v has a trivalent neighbour by Property 18. Thus $\delta(H_v) = 2$ and $H_v \cong Ch_k$ for some $k \geq 2$ or $H_v \cong SCh_\ell$ for some $\ell \geq 1$. By the recursive constructions of Ch_k and SCh_ℓ we get that either $H \cong Ch_{k+1}$ or $H \cong SCh_{\ell+1}$. \square

Property 23. If $H \leq G$ is connected, $\nu(H) \leq 0$, $\delta(H) = 3$ and there is a trivalent vertex $v \in V(H)$ such that $N(C_4; v) > 0$, then $H \cong BC_k$ for some $k \geq 5$.

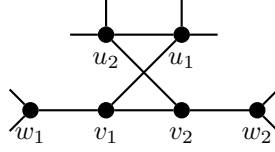
Proof. Suppose that $a \in V(H)$ is a trivalent vertex such that $N(C_4; a) > 0$, say $\{a, b, c, d\} \subseteq V(H)$ forms a cycle of length four in H , where $ab, ad \in E(H)$. Since $\delta(H) = 3$ we have that $d^2(a) \geq 9$. Thus $\nu(H_a) \leq \nu(H) + 6 - (1) \leq 5$.

By Property 9 the vertex c lies in a K_2 -component of H_a if c is trivalent in H . In this case say that t is the other vertex of that K_2 -component. Then t must have at least two common neighbours with a (since $d(t) \geq 3$), whence $N(t) \cap N(c) \neq \emptyset$, contradicting H being triangle-free.

Hence there are no 4-cycles in H with trivalent vertices opposite to each other (i.e. at distance two in the cycle). If both b and d were at least tetravalent, then $d^2(a) \geq 11$. This would give $\nu(H_a) \leq \nu(H) + 0 - (1) \leq -1$, contradicting assumption (A1). Suppose, on the other hand, that $d(a) = d(b) = 3$ but $d(d), d(c) \geq 4$. Then $d(d) = d(c) = 4$ by a similar argument to the previous (either $\nu(H_a) \leq -1$ or $\nu(H_b) \leq -1$, in both cases contradicting assumption (A1)).

If $v_1 \in V(H)$ is a trivalent vertex such that $N(C_4; v_1) > 0$ we must therefore have that the local structure in the neighbourhood of v_1 is as in Figure 16. We let v_2 be the trivalent neighbour of v_1 in a 4-cycle. Let u_i be the tetravalent neighbour of v_i in the cycle, and define w_i to be the trivalent neighbour of v_i in $N(v_i) \setminus \{u_i, v_{3-i}\}$.

Suppose that $d^2(u_1) \geq 15$. Then $\nu(H_{u_1}) \leq \nu(H) + 5 - (1) \leq 4$ and $d(H_{u_1}; v_2) = 1$. Thus $d(H_{u_1}; w_2) = 1$ by Property 9. Therefore u_1 and w_2 has two common neighbours. Since $u_1, v_1 \notin N(w_2)$ we get $|N(w_2) \cap N(u_1)| = |N(u_1) \setminus \{v_1, u_1\}| = 2$ and $N(C_4; u_1) \geq 2$. But then $\nu(H_{u_1}) \leq \nu(H) + 5 - (2) \leq 3$, and $d(H_{u_1}; v_2) = 1$, contradicting Property 4.

FIGURE 16. The local structure around the trivalent vertex v_1 in H .

Hence, $d^2(u_1) \leq 14$ and therefore, in particular, u_1 has at least two trivalent neighbours. Analogously we get that $d^2(u_2) \leq 14$.

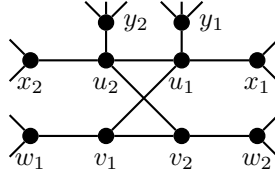
Let $x_i, y_i \in N(u_i) \setminus \{v_i, u_{3-i}\}$ be such that $d(x_i) \leq d(y_i)$ for $i \in [2]$. Then, in particular, $d(x_i) = 3$.

Suppose that $d(y_1) = 3$, then $\nu(H_{v_1}) \leq \nu(H) + 3 - (1) \leq 2$. If H_{v_1} is connected then $H_{v_1} \cong C_5$ by Property 22 and since there are more than four bivalent vertices in H_{v_1} (five or six depending on whether w_1 has two or three trivalent neighbours). Then u_1 would have three neighbours in a cycle of length five, contradicting H being triangle-free.

Hence, H_{v_1} is not connected. By Property 22 and since $|N_2(v_1)| \leq 6$ we have no more than two components, whence $|\mathcal{C}(H_{v_1})| = 2$.

None of the components of H_{v_1} can contain all the bivalent vertices since then it would contain at least five bivalent vertices, and then we would have a C_5 -component in which u_1 has three neighbours. Therefore both components have ν -value at most two and minimum valency at most two, as well. So by Property 22 we have that each of the two components contain 1, 2, 4 or 5 bivalent vertices. Hence the distribution of bivalent vertices among the two components is either 1 + 5 (in which case $H_{v_1} \cong Sch_1 + C_5$) or 2 + 4 (in which case $H_{v_1} \cong Sch_{\geq 2} + Ch_{\geq 3}$). However, $H_{v_1} \cong Sch_1 + C_5$ is not possible since then u_1 would have two neighbours in a C_5 -component, giving $N(C_4; N(v_1)) \geq 2$. This would in turn yield $\nu(H_{v_1}) \leq 1$, but this contradicts $\nu(Sch_1 + C_5) = 2$. On the other hand, also $H_{v_1} \cong Sch_{\geq 2} + Ch_{\geq 3}$ is impossible because the four bivalent vertices of the $Ch_{\geq 3}$ -component must all be in $N_2(v_1)$. Hence, $|N_2(v_1) \cap V(Sch_{\geq 2})| \leq 2$, but $Sch_{\geq 2}$ is 2-stable by Property 6. This would therefore give redundant edges in H by Lemma 11, and therefore contradict Property 2.

This shows us that $d(y_1) \geq 4$, and since $d^2(u_1) \leq 14$ we get that $d(y_1) = 4$. Analogously, $d(y_2) = 4$. So the situation looks like in Figure 17.

FIGURE 17. The local structure around the trivalent vertex v_1 in H .

Note that in H_{v_1} the vertices w_2 , x_1 and u_2 are bivalent, and moreover so is at least one of the vertices in $N(w_1) \setminus \{v_1\}$.

Suppose that $e(\{x_2, y_2\}, N[v_1]) = 0$. Then $d(H_{v_1}; u_2) = 2$, $d^2(H_{v_1}; u_2) = 3 + 4 = 7$, which is impossible since then $\nu(H_{v_1, u_2}) \leq \nu(H) + 3 - (1) - 5 \leq -3$. Hence, $e(\{x_2, y_2\}, N[v_1]) \geq 1$. Neither y_2 nor x_2 is adjacent to v_1, u_1 or v_2 . Thus, we must have that $y_2 w_1 \in E(H)$ or $x_2 w_1 \in E(H)$.

Now, suppose that $e(\{x_2, y_2\}, w_1) = 1$. We then have that $\nu(H_{v_1, u_2}) \leq \nu(H) + 3 - (1) - 2 \leq 0$ since $d^2(H_{v_1}; u_2) = 6$. Also, $d(H_{v_1, u_2}; x_1) = 2$ and therefore $x_2 \in C \cong Ch_k \in \mathcal{C}(H_{v_1, u_2})$ for some $k \geq 2$.

If $k = 2$ then the five vertices in $N(v_1) \cup N(u_2)$ that may be adjacent to some vertex of $V(C)$ are w_1, v_2, u_1, y_2 and x_2 . If any of them were adjacent to more than one vertex in $V(C)$ we would have $N(C_4; N(v_1) \cup N(u_2)) \geq 2$, giving $\nu(H_{v_1, u_2}) \leq -1$, contradicting assumption (A1). Hence $e(x, V(C)) = 1$ for all $x \in \{w_1, v_2, u_1, y_2, x_2\}$, and therefore $w_2 \in V(C)$. Say that $\{\alpha, \beta\} = V(C) \setminus N(x_1)$. We now have $\nu(H_{x_1}) \leq \nu(H) + 3 - (1) \leq 2$ and $d(H_{x_1}; \alpha) = d(H_{x_1}; \beta) = 2$. If $d^2(H_{x_1}; \alpha) = d^2(H_{x_1}; \beta) = 2 + 3$, then there is a cycle of length four through the trivalent neighbours of α and β in H_{x_1} by Property 15. We then would have that $N(C_4; N(v_1) \cup N(u_2)) \geq 2$, which would yield $\nu(H_{v_1, u_2}) \leq -1$, contradicting the inductive assumption. Thus, both α and β have second valency $2 + 4$ (by Property 18), in H_{x_1} . Both α and β are then adjacent to y_2 since y_2 and u_1 are the only tetravalent vertices among w_1, v_2, u_1, y_2, x_2 and $u_1 \notin V(H_{x_1})$. This, however, contradicts H being triangle-free.

Hence, $k \geq 3$. If $e(N[v_1], N(x_1) \setminus \{u_1\}) = 0$, then $d^2(H_{v_1}; x_1) = 6$ which implies that $\nu(H_{v_1, x_1}) \leq \nu(H) + 3 - (1) - 2 - (1) \leq -1$. Therefore we must have that $e(N[v_1], N(x_1) \setminus \{u_1\}) \geq 1$. Now, $d^2(x_1) = 4 + 3 + 3$ and $N(C_4; N(x_1)) \geq 2$, so $\nu(H_{x_1}) \leq \nu(H) + 3 - (2) \leq 1$. Moreover, $d(H_{x_1}; v_1) = 2$ which implies that $v_1 \in C' \cong Ch_\ell \in \mathcal{C}(H_{x_1})$ for some $\ell \geq 2$. Therefore at least one of the vertices w_1 and v_2 is adjacent to $N(x_1) \setminus \{u_1\}$. If not both of them are, then $\ell \geq 3$ and the trivalent neighbour of v_1 in H_{x_1} has a 4-cycle through it. This would give us that $N(C_4; N(v_1)) \geq 2$, whence $\nu(H_{v_1, u_2}) \leq \nu(H) + 3 - (2) - 2 \leq -1$, contradicting assumption (A1). We must therefore have that both w_1 and v_2 are adjacent to $N(x_1) \setminus \{u_1\}$, which gives us that $C' \cong C_5$. But $d(x_1) = 3$, so at least two vertices in $N(x_1)$ has two neighbours in $V(C')$.

If $y_2 \in V(C')$ then since y_2 is tetravalent all three neighbours of x_1 would have to have two neighbours in C' , giving $\nu(H_{x_1}) \leq \nu(H) + 3 - (3) \leq 0$, and $x_2 \in N(x_1)$, $u_2 \in V(C')$. Both y_2 and u_2 are tetravalent in H but bivalent in H_{x_1} , so $e(C', H \setminus C') \geq 7$, and therefore at least one vertex of $N(x_1)$ would have to be adjacent to three vertices in $V(C')$. This contradicts H being triangle-free.

Hence we must have $y_2 \notin V(C')$. This means, in particular, that $w_1 x_2 \in E(H)$ and $V(C') = \{v_1, v_2, u_2, x_2, w_1\}$. However, this would make $y_2 \in N(x_1)$, contradicting $d^2(x_1) = 3 + 3 + 4$.

Since all cases when $e(\{x_2, y_2\}, w_1) = 1$ lead to contradictions we must have that $e(\{x_2, y_2\}, w_1) = 2$ and analogously we get $e(\{x_1, y_1\}, w_2) = 2$. The situation is therefore as in Figure 18.

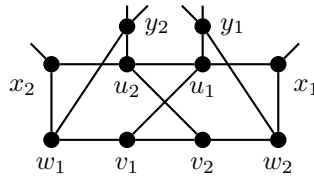


FIGURE 18. The local structure around the trivalent vertex v_1 in H .

Note that in particular in this situation we have that $N(C_4; N(v_1)) = 3$ and $N(C_4; u_2) \geq 2$. This means that we get $\nu(H_{v_1}) \leq \nu(H) + 3 - (3) \leq 0$. If H_{v_1} were disconnected it would have to contain either at least eight bivalent vertices, or at

least four bivalent vertices and a 3-stable component, in either case contradicting $|N_2(v_2)| = 6$.

Since (x_1, w_2) and (x_2, u_2) are pairs of adjacent bivalent vertices this makes H being formed from some Ch_k -component by adding the vertices v_1, v_2, w_1 and u_2 and edges as illustrated in Figure 19.

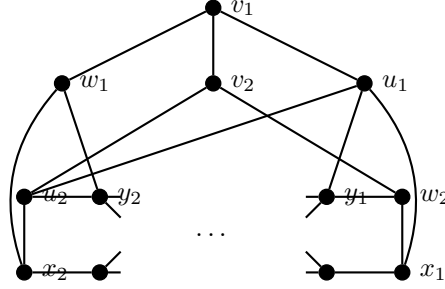


FIGURE 19. How v_1 and v_2 's neighbourhood looks in relation to the Ch_k -component in H_{v_1} .

It is easily seen that $k \geq 4$ since H should be triangle-free and $N(C_4; N(v_1)) = 3$. This makes $H \cong BC_k$ for some $k \geq 5$ since $(BC_k)_v \cong Ch_{k-1}$ where v is any trivalent vertex (connected by edges as indicated in Figure 19). \square

We will now show that connected subgraphs of G with ν -value zero and minimum valency three either have no cycles of length four or are isomorphic to BC_k s. We begin by showing that there cannot be cycles of length four through vertices of low valency. We later use this to exclude such cycles altogether. We begin with neighbourhoods of vertices of valency three.

Property 24. If $H \leq G$ is connected, $\nu(H) \leq 0$, $\delta(H) = 3$ and $H \not\cong BC_k$ for all $k \geq 5$, then $N(C_4; N(v)) = 0$ for all trivalent vertices v of H .

Proof. Suppose otherwise, i.e. there is some trivalent vertex $v \in V(H)$ such that $N(C_4; N(v)) > 0$. By Property 23 the vertex v must have at least one non-trivalent neighbour. Hence, $d^2(v) \geq 10$, and clearly $d^2(v) \leq 10$ since otherwise $\nu(H_v) \leq \nu(H) + 0 - (1) \leq -1$, contradicting assumption (A1).

We therefore have $\nu(H_v) \leq \nu(H) + 3 - (1) \leq 2$. Let w_1, w_2 denote the two trivalent neighbours of v . If we had that $d^2(w_i) = 11$ for both $i \in [2]$ then we would get $\nu(H_{w_1}) \leq \nu(H) + 0 \leq 0$ and $d(H_{w_1}; w_2) = 2$. Thus, w_2 would belong to a Ch_k -component of H_{w_1} for some $k \geq 2$. It is clear that $k \neq 2$ since otherwise the two tetravalent neighbours of w_1 would have to have two common neighbours with w_2 . On the other hand, for $k \geq 3$ we have that $N(C_4; N(w_1)) \geq 1$ and we would then get $\nu(H_{w_1}) \leq -1$, contradicting assumption (A1).

Hence, not both w_1 and w_2 can have second valency eleven. There is therefore bivalent vertices in H_v . If there is only one bivalent then, by Property 22, we must have that SCh_1 is a component of H_v . But SCh_1 contains a cycle of length four containing only trivalent vertices. All four of these vertices must have valency at least four in H by Property 23. There are only three neighbours of v in H , so at least one of them is adjacent to two of the trivalent vertices in the cycle of length four. But then we get $N(C_4; N(v)) \geq 2$, which would give $\nu(H_v) \leq 1$. However $\nu(SCh_1) = 2$ so we would have to have another component of H_v with negative ν -value, contradicting assumption (A1).

Thus there are at least two bivalent vertices of H_v and therefore, by Property 22, we must have either a C_5 in $\mathcal{C}(H_v)$ or at least eight vertices with different valency in H and H_v . The latter is impossible since $|N_2(v)| = 7$. The former is impossible since we would get that at least one of w_1 and w_2 has two neighbours in a C_5 -component. This would yield a 4-cycle through a trivalent vertex, contradicting Property 23. \square

Property 25. If $H \leq G$ is connected, $\nu(H) \leq 0$, $\delta(H) \geq 3$, $H \not\cong BC_k$ for all $k \geq 5$, then $N(C_4; v) = 0$ for all tetravalent vertices v of H .

Proof. Suppose, to the contrary, that there is a cycle, C , of length four in H containing a tetravalent vertex v . Note that all the vertices in C have valency four since otherwise there would be a tetravalent vertex $u \in V(C)$ with $d^2(u) \geq 17$ by Property 24. This is not possible since then $\nu(H_u) \leq \nu(H) - 1 - (1) \leq -2$, contradicting assumption (A1).

Hence, $d^2(v) = 16$ and therefore $\nu(H_v) \leq \nu(H) + 2 - (1) \leq 1$. Let $x \in C$ be the vertex at distance two from v in C . Since x has two common neighbours with v we have $d(H_v; x) = 2$. One of the two vertices in $N(x) \setminus V(C)$ has two neighbours in $N(x) \setminus V(C)$, and the other has at least one. This follows from $d^2(H_v; x) \leq 5$ (by Property 16) and from H being triangle-free. Thus $N(C_4; N(v)) \geq 3$, which implies that $\nu(H_v) \leq \nu(H) + 2 - (3) \leq -1$, contradicting assumption (A1). \square

Property 26. If $H \leq G$ is connected, $\nu(H) \leq 0$ then $N(C_4; v) = 0$ for all pentavalent vertices v of H .

Proof. Suppose otherwise, i.e. that there is a cycle of length four, C on $\{c_1, c_2, c_3, c_4\}$ where $c_1c_3 \notin E(H)$, containing a pentavalent vertex, say c_1 . Since BC_k, Ch_ℓ contains no pentavalent vertices we have $H \not\cong BC_k$ and $H \not\cong Ch_\ell$ for all $k \geq 5$ and all $\ell \geq 2$.

Clearly $\delta(H) \geq 3$ then by Properties 4 and 16. All vertices in C are at least pentavalent by Property 25. Since c_1 has no trivalent neighbours (by Property 24) and at least two neighbours of valency at least five we have $d^2(v) \geq 2 \cdot 5 + 3 \cdot 4 = 22$. This gives $\nu(H_v) \leq \nu(H) + 1 - (1) \leq 0$. Hence, in particular, all the vertices of C are pentavalent with three tetravalent and two pentavalent neighbours.

Let $M_i = N(c_i) \setminus V(C)$ for $i \in [4]$. We have that $e(M_i, M_{i+2}) \geq 2$ for $i \in [2]$ since otherwise c_{i+2} would be trivalent with second valency at least eleven in H_{c_i} . We would moreover have that $\nu(H_{c_i}) \leq 0$. Thus by assumption (A1) we get that all components of H_{c_i} are in \mathcal{G} , but in all of the graphs of \mathcal{G} the trivalent vertices all have second valency at most ten.

Each of the vertices in M_i (for $i \in [4]$) must have at least one trivalent neighbour, since otherwise there would be some vertex $m \in M_i$ such that $d^2(m) \geq 17$. This gives us a contradiction to assumption (A1) by considering H_m . Thus, a fortiori, H_{c_1} contains some bivalent vertices, and therefore at least four bivalent vertices by assumption (A1). Note also that $|N_2(c_1)| \leq 16$.

All trivalent vertices, except possibly c_3 , in H_{c_1} must belong to W_5 - or $(2C_7)_{2i}$ -components since otherwise there would be a cycle of length four through such a vertex. That vertex would then have to be at least pentavalent in H by Properties 23 and 25. This would yield $N(C_4; N(c_1)) \geq 2$, whence $\nu(H_{c_1}) \leq -1$, contradicting assumption (A1).

Hence, $\mathcal{C}(H_{c_1})$ consists of C_5 s, W_5 s and $(2C_7)_{2i}$ s, at least one of which is a C_5 . Now, since $d(H_{c_1}; x) = 3$ for all $x \in M_2 \cup M_4 \cup \{c_3\}$, at least one of the vertices in M_1 must have two neighbours in the C_5 -component. This gives us that $N(C_4; N(c_1)) \geq 2$, and therefore $\nu(H_{c_1}) \leq -1$, contradicting assumption (A1). \square

We can now conclude this study of subgraphs with ν -value zero and cycles of length four by the following property. It claims that the only way to have cycles of length four in a graph with ν -value zero is to be one of the known graphs with 4-cycles in \mathcal{G} .

Property 27. If $H \leq G$ is connected, $\nu(H) \leq 0$ then exactly one of the following three statements holds.

- (i) $H \cong Ch_k$ for some $k \geq 2$.
- (ii) $H \cong BC_k$ for some $k \geq 5$.
- (iii) $N(C_4; H) = 0$ and $H \not\cong C_5$.

Proof. Suppose that neither (i) nor (ii) is true. Then we must have $\delta(H) \geq 3$ by Properties 4 and 16.

We will prove that no cycle of length four contains a vertex of valency d for all $d \geq 3$ by induction on d . This holds for $d = 3, 4, 5$ by Properties 23, 25 and 26. Suppose therefore that $d \geq 6$ and that there are no cycles of length four through any vertices with valency less than d .

Now, suppose that $v \in V(H)$ has valency d and is such that $N(C_4; v) > 0$. Then $d^2(v) \geq 2 \cdot d + (d-2) \cdot 4$ since its two neighbours in the cycle of length four would have valency at least d and by Property 24 the remaining $d-2$ neighbours have valency at least four. Then by Property 1 we get $\nu(H_v) \leq \nu(H) - 3d^2(v) + 17d - 18 - (1) \leq \nu(H) - d + 5 \leq -1$, contradicting assumption (A1). The conclusion now follows by induction. \square

Property 28. If $H \leq G$ is connected, $\nu(H) = 3$, $\alpha(H_2) > 1$ and $N(C_4; H) = 0$, then there is an edge $e \in E(H)$ such that $H - e = C_5 + H'$, where $H' \in \{C_5, W_5, (2C_7)_{2i}\}$.

Proof. Suppose first that H is not edge-critical, then there is an edge $e \in E(H)$ such that $\nu(H - e) = 0$. $H - e$ contains vertices that are at most bivalent so it contains a C_5 -component. It can clearly not be the only component. Hence $H - e = C_5 + H'$ where $\nu(H') = 0$, and the conclusion follows from assumption (A1).

On the other hand, if H is edge-critical we get the following. Let $v \in V(H)$ be some bivalent vertex. We must have that $d^2(v) \leq 5$, otherwise $\delta(H_v) \leq 2$ (since $\alpha(H_2) > 1$) and $|\mathcal{C}(H_v)| = 1$ (by Lemma 5) which would imply that $H_v \cong C_5$, because $\nu(H_v) \leq 1$. We would then get several cycles of length four through the neighbourhood of v , yielding a contradiction to assumption (A1).

Since H is not 2-regular there is a bivalent vertex v_1 in H with a trivalent neighbour. Suppose the bivalent neighbour of v_1 , say v_2 , does not have second valency five. Let $\{u\} = N(v_2) \setminus \{v_1\}$, $\{t_1\} = N(v_1) \setminus \{v_2\}$ and $\{u', t_2\} = N(t_1) \setminus \{v_1\}$. See Figure 20 for an illustration of the situation.

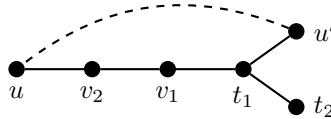


FIGURE 20. The situation in the neighbourhood of v_1 and v_2 in the case that v_2 does not have second valency 5.

Now, u belongs to a P_2 -component of H_{v_1} by Property 9, so the edge $t_1 t_2 \in E(H)$ would be redundant since the component, C , containing t_2 in H_{v_1} would have $\nu(H) = 0$ and thus be 2-stable by Property 6. This contradicts the edge-criticality of H .

Hence, we have that H_2 only consists of P_2 -components. Furthermore H_2 has at least two such components since $\alpha(H_2) > 1$. Say that a_1 and a_2 forms another such component and give names to the vertices in their neighbourhoods according to Figure 21. Note that the vertices x_i and the vertices c_i need not necessarily be distinct.

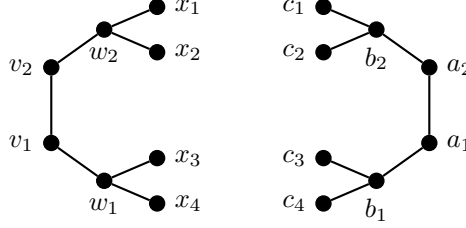


FIGURE 21. Names of the vertices in the H -neighbourhoods of the two P_2 -components of H_3 .

If $d(x_1) = 2$ then $d(H_{v_2}; x_1) \leq 1$ so x_1 would belong to a P_2 -component of H_{v_2} , while x_2 and w_1 would belong to components that have ν -value zero and therefore belong to \mathcal{G} by assumption (A1). This is not possible however, since by Property 6 these components are 2-stable.

Hence,

$$(4) \quad d(x_i), d(c_i) \geq 3 \quad (\forall i \in [4]).$$

Suppose that $d^2(H_{v_1}; w_2) \geq 6$. Then $\nu(H_{v_1, w_2}) \leq 2$ so a_1 and a_2 belong to a C_5 -component, say C , of H_{v_1, w_2} . All vertices of C except for a_1 and a_2 have valency at least three in H and so each of the vertices w_1 , x_1 and x_2 must have a neighbour in $V(C)$. This leaves three other edges in $E(N[\{v_1, w_2\}], V(H) \setminus N[\{v_1, w_2\}])$, so if $C' \in \mathcal{C}(H_{v_1, w_2}) \setminus \{C\}$ we must have that $\delta(C') \leq 2$ or otherwise C' would be 3-stable by Property 11. But then also $C' \cong C_5$, whence $H_{v_1, w_2} \cong C_5 + C_5$, which gives $\nu(H_{v_1, w_2}) = 0$. This would however require that $N(C_4; N(v_1) \cup N(w_2)) \geq 2$, which contradicts $N(C_4; H) = 0$.

Hence, $d^2(H_{v_1}; w_2) \leq 5$ and analogously $d^2(H_{v_2}; w_1) \leq 5$.

Now, by (4) there must therefore be a cycle of length five through v_1 and v_2 . Analogously we get one through a_1 and a_2 . We may therefore assume that the situation looks like illustrated in Figure 22.

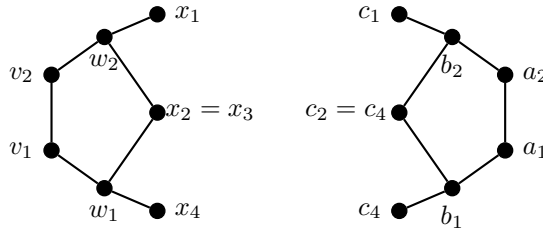


FIGURE 22. The H -neighbourhoods of the two P_2 -components of H_3 .

Clearly $x_1 \neq x_4$ since we have no cycles of length four. We also have $d(x_1) = 3$ since $d^2(H_{v_1}; w_2) \leq 5$. Analogously we have $d(x_4) = d(c_1) = d(c_4) = 3$. In a similar manner we see that $d(x_2) = d(c_2) = 3$.

We have that $d^2(x_1) \leq 9$ since otherwise $\nu(H_{x_1, v_2}) \leq \nu(H) + 3 - 7 \leq -1$, contradicting assumption (A1). Suppose that $d^2(x_1) = 9$. We then instead get that $\nu(H_{x_1, v_2}) = \nu(H) + 6 - 7 = 2$. The vertices a_1 and a_2 are at most bivalent in H_{x_1, v_2} and must therefore belong to some C_5 -component, C , of H_{x_1, v_2} . The same goes for the vertex w_1 as well. If there are two C_5 -components in H_{x_1, v_2} then these are the only two components since $e(N[\{x_1, v_2\}], V(H) \setminus N[\{x_1, v_2\}]) \leq 6$. But then $\nu(H_{x_1, v_2}) = 0$, which would give us that $N(C_4; N(a) \cup N(v_2)) = 2$, contradicting that H contains no cycles of length four. Thus a_1, a_2 and w_1 must belong to the same C_5 -component, i.e. we must have the situation illustrated in Figure 23 (note in particular that x_1 and x_4 must have a common neighbour).

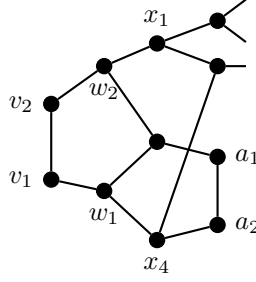


FIGURE 23. The neighbourhood of v_1, v_2, a_1, a_2 in the case that $d^2(x_1) = 9$.

The three remaining edges in $E(N[\{x_1, v_2\}], V(H) \setminus N[\{x_1, v_2\}])$ must go to a component C' such that $\delta(C') \leq 2$ by edge-criticality and Property 11. But then $C' \cong C_5$ and we would have two C_5 -components, which we already have seen to be impossible.

Hence, $d^2(x_1) \leq 8$. This means in particular that x_1 has some bivalent neighbour, t_1 , which in turn must have a bivalent neighbour t_2 , in H . t_1 and t_2 are both monovalent in H_{v_1, w_2} (by Property 9) since $\nu(H_{v_1, w_2}) = \nu(H) + 1 + 1 = 5$. Clearly $t_2 w_1 \notin E(H)$ so we must have that $t_2 x_2 \in E(H)$. Since $d(x_2) = 3$ we get that $H_{v_1} \setminus \{t_1, t_2\}$ is a graph with ν -value one. There are then only two edges in $E(T, H \setminus T)$ where $T = \{v_1, v_2, w_1, w_2, x_2, x_1, t_1, t_2\}$. This is not possible however since $H_{v_1} \setminus \{t_1, t_2\}$ is 2-stable (by Property 6). \square

Corollary 3. Let $H \leq G$ be connected, $\nu(H) \leq 0$ and $N(C_4; H) = 0$. If $v \in V(H)$ is trivalent with $d^2(v) \geq 10$, then either $\delta(H_v) \geq 3$ or $\alpha((H_v)_2) = 1$.

Proof. If $d^2(v) \geq 11$, then $\nu(H_v) \leq \nu(H) + 0 \leq 0$. By assumption (A1) we would then get that either $\delta(H_v) \geq 3$ or H_v contains C_5 -components. It is however easily seen that H_v can not contain C_5 -components since $N(C_4; H) = 0$.

Hence, $d^2(v) = 10$. We must have that H_v contains a least one bivalent vertex. If $\nu(H) < 0$ then $\nu(H_v) \leq 2$ and the conclusion follows from Property 22. On the other hand if $\nu(H) = 0$, then $\nu(H) = 3$ and by Property 28 H_v has at least four bivalent vertices in a cycle of length five if $\alpha((H_v)_2) > 1$. But if this were the case then one of v 's three neighbours would have to be adjacent to two vertices in a cycle of length five. This would contradict that $N(C_4; H) = 0$. \square

Property 29. Suppose that G is connected, $\nu(G) \leq 0$ and $\delta(G) = 3$. If $G \notin \{BC_k; k \geq 5\} \cup \{W_5, (2C_7)_{2i}\}$ then G_3 is 2-regular.

Proof. Firstly, $N(C_4; G) = 0$ by Proposition 27. G is not 3-regular by Property 8. Note that $\delta(G_3) \geq 1$ since otherwise there would be a trivalent vertex v in G with

second valency at least twelve. This would however give $\nu(G_v) \leq \nu(G) - 3 \leq -3$, contradicting assumption (A1).

Now, suppose that $v \in V(G_3)$ is monovalent in G_3 , then we instead get $\nu(G_v) \leq 0$. By assumption (A1) we therefore have that all components of G_v are in $\{C_5, W_5, (2C_7)_{2i}\}$. We can easily see that it is not possible to have C_5 -components in G_v since then one of v 's neighbours would have to be adjacent to two vertices in that C_5 -component, which would give a cycle of length four. So all components of G_v are isomorphic to one of the two 3-regular components W_5 and $(2C_7)_{2i}$. The trivalent neighbour, u , of v in G would then have two tetravalent neighbours. Let w_{11}, w_{12} denote the two tetravalent neighbours of v , and w_{21}, w_{22} the two tetravalent neighbours of u . Analogously as for G_v we can show that all components of G_u must belong to $\{W_5, (2C_7)_{2i}\}$. This means, in particular, that the neighbours of w_{ij} that are not u or v must be tetravalent in G for all $i, j \in [2]$. But then we must have that all three vertices in $N(w_{11}) \setminus \{v\}$ are adjacent to w_{21} or w_{22} . This would however yield a cycle of length four through w_{11} , contradicting $N(C_4; G) = 0$.

Hence we must have $\delta(G_3) \geq 2$. Suppose now that there is a bivalent vertex v in G_3 with a trivalent neighbour, u . By Corollary 3 we must have that $\alpha((G_v)_2) = 1$, which would make the two neighbours of u that are not v adjacent, contradicting that G is triangle-free. \square

Property 30. If G is connected, $\nu(G) \leq 0$, $N(C_4; G) = 0$, $\delta(G) = 3 \neq \Delta(G)$, then $G_3 \cong \left(\frac{|V(G_3)|}{5}\right) \cdot C_5$.

Proof. By Property 29 the induced graph G_3 consists of 2-regular components. Clearly G_3 does not contain any C_4 -components. If G_3 were to contain a cycle of length six or more, then let v be a vertex of that cycle. We would then have $\delta(G_v) \leq 2$ with $\alpha((G_v)_2) > 1$, contradicting Corollary 3. \square

Property 31. Suppose that G is connected and $\nu(G) \leq 0$. If $G \notin \mathcal{G}$ then G is 4-regular.

Proof. Note that $N(C_4; G) = 0$ by Property 27. We have that $\delta(G) \geq 3$ by Property 4. Moreover, we can conclude that $\Delta(G) \neq 3$ from Property 8. Thus, Property 30 implies that G_3 consists only of C_5 -components (possibly none).

Suppose that $V(G_3)$ is non-empty, i.e. that $\delta(G) = 3$. We then have a cycle of length five on the vertices v_1, v_2, v_3, v_4, v_5 in $V(G_3)$. The neighbours of these vertices that are not in the cycle, say w_i is adjacent to v_i , must all be tetravalent in G . The situation is therefore as illustrated in Figure 24.

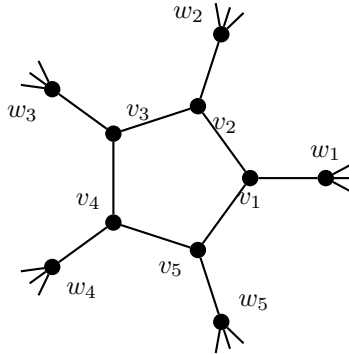


FIGURE 24. The neighbourhood of the C_5 from G_3 in G .

Clearly, $\text{dist}(w_i, w_{i+1}) \geq 2$ for all $i \in [5]$ (taking indices modulo 5). On the other hand if $\text{dist}(w_1, w_2) = 3$, then consider the graph $G' := G \setminus (\{v_i; i \in [5]\} \cup \{w_4\}) + w_1w_2$. We must have that G' is triangle-free, $n(G') = n(G) - 6$, $e(G') = e(G) - 12$ and $N(C_4; G') \leq 3$ since $e(N(w_1) \setminus v_1, N(w_2) \setminus v_2) \leq 3$. Moreover, $\alpha(G') < \alpha(G)$ since any independent set of G' contains at most one of w_1 and w_2 and may therefore be extended to an independent set of size one more in G .

Similarly, any independent set of size $\alpha(G) - 1$ in $G \setminus \{v_i; i \in [5]\}$ must contain at least three consecutive w_j 's, or could otherwise be extended to an independent set of size $\alpha(G) + 1$ in G . The same goes for $G'' := G \setminus \{v_i; i \in [5]\} + w_1w_2$ since $\alpha(G'') \leq \alpha(G \setminus \{v_i; i \in [5]\})$. So in particular any independent set in G'' of size $\alpha(G) - 1$ must contain w_4 , since it contains at most one of w_1 and w_2 . Hence, $\nu(G') \leq \alpha(G) - 2$, which gives $\nu(G') \leq \nu(G) - 3 \cdot 12 + 17 \cdot 6 - 35 \cdot 2 + (3) \leq -1$, contradicting assumption (A1).

Therefore, $\text{dist}(w_1, w_2) = 2$ and by analogous arguments we have $\text{dist}(w_i, w_{i+1}) = 2$ for all $i \in [5]$.

Suppose that $w_1w_3 \notin E(G)$ and that $\delta(G_{v_1, v_3}) \leq 2$. Then we would have $\nu(G_{v_1, v_3}) \leq \nu(G) + 3 - 2 \leq 1$, so G_{v_1, v_3} contains a C_5 -component. w_1 and w_3 are adjacent to at most one vertex in such a component, each. The remaining three vertices must then be w_2, w_4 and w_5 . These vertices, however, are at least tetravalent in G , so all of them would have to be adjacent to w_1 or w_2 , which would give a cycle of length three or four. This, and analogous arguments, gives us that $w_iw_{i+2} \notin E(G)$ implies $\delta(G_{v_i, v_{i+2}}) \geq 3$.

We must have that either $w_iw_{i+2}, w_iw_{i-2} \in E(G)$ or $w_iw_{i+2}, w_iw_{i-2} \notin E(G)$ for all $i \in [5]$, since otherwise we may without loss of generality assume that $w_1w_3 \notin E(G)$ but $w_1w_4 \in E(G)$. In this case we would get $\nu(G_{v_1, v_3}) \leq 1$ but this would make w_4 bivalent in G_{v_1, v_3} , contradicting the previous.

Suppose that $w_iw_{i+2} \in E(G)$ for some $i \in [5]$. Then we would have to have $w_jw_{j+2} \in E(G)$ for all $j \in [5]$, i.e. the local structure is as illustrated in Figure 25.

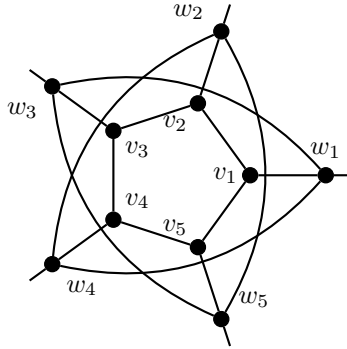


FIGURE 25. The neighbourhood of the C_5 from G_3 in G with edges between w_i s.

In this case it is impossible to have three consecutive w_i s in an independent set of $G \setminus \{v_i; i \in [5]\}$. This means that $\alpha(G \setminus \{v_i; i \in [5]\}) \leq \alpha(G) - 2$, which would make all the v_iw_{i-2} edges redundant. Therefore we must have that $w_iw_{i+2} \notin E(G)$ for all $i \in [5]$.

If $d^2(w_i) \geq 16$ for some $i \in [5]$ then v_i is the only trivalent neighbour of w_i . We would then have $\nu(G_{w_i}) \leq \nu(G) + 2 \leq 2$, with v_{i-2} and v_{i+2} bivalent. It is easy to see that this is not possible since then they would have to belong to a

C_5 -component. We must therefore have that $d^2(w_i) = 3 + 4 + 4 + 4 = 15$ for all $i \in [5]$.

Suppose that $d^2(G_{w_1, v_2; v_4}) = 6$. Then $\nu(G_{w_1, v_2, v_4}) \leq 1$, with w_5 bivalent (since $N(w_1) \cap N(w_5) \neq \emptyset$ and $N(v_4) \cap N(w_5) = \{v_5\}$). This implies that w_5 belongs to a C_5 -component, C , of G_{w_1, v_2, v_4} . By the previous, however, $V(C)$ contains at least three vertices that are tetravalent in G . Hence, $e(C, N(w_1) \cup N(v_2) \cup N(v_4) \setminus \{v_1\}) \geq 8$, but $|N(w_1) \cup N(v_2) \cup N(v_4) \setminus \{v_1\}| \leq 7$, which would therefore give a cycle of length four in G . Hence, $d^2(G_{w_1, v_2; v_4}) \leq 5$. Since $e(w_4, N(v_2)) = 0$ we get $N(w_1) \cap N(w_4) \neq \emptyset$. Completely analogously we may show that $N(w_i) \cap N(w_{i+2}) \neq \emptyset$ for all $i \in [5]$.

Now, note that $G_3 = \emptyset$, since $\nu(G_{v_1, v_3}) \leq \nu(G) + 3 - 2 \leq 1$, and the common neighbour of w_1 and w_3 would have valency 2 in G_{v_1, v_3} , contradicting that $w_i w_{i+2} \notin E(G)$ implies $\delta(G_{v_i, v_{i+2}}) \geq 3$ for all $i \in [5]$.

Thus $\delta(G) \geq 4$ and G is 4-regular by Property 5 and since $d^2(v) \leq 16$ for all $v \in V(G_4)$. \square

3. PROPERTIES OF A MINIMAL COUNTEREXAMPLE

We will from here on assume, for a contradiction, that G is a minimal counterexample to the assertion in Theorem 1, i.e. we assume that either $\nu(G) < 0$ or $\nu(G) = 0$ but $G \notin \mathcal{G}$ but still assuming (A1). If $\nu(G) < 0$ then we in fact have $\nu(G) \leq -1$ since ν only takes integer values. In particular, G must be a connected graph. Note that this is just a strengthening of assumption (A1), so all the properties for G derived thus far holds also under this stronger assumption on G .

Note that by Property 27 we must have $N(C_4; G) = 0$. Also, G is 4-regular by Property 31.

3.1. Graphs with valencies three and four and ν -value two. A lot of things in this section are quite close to the works of Radziszowski and Kreher in [9] and the slight modification by Backelin in [2]. Some of the following results are just reformulations of their results in our particular context.

Lemma 15. If H is a graph such that $\delta(H) = 3$, $\Delta(H) \leq 4$, $\nu(H) \leq 2$ and $H \leq G$, then $\delta(H_3) \geq 1$, and every $v \in V(H)$ such that $d(H_3; v) = 1$ belongs to a K_2 -component of H_3 and has no trivalent vertices at distance two from v in H .

Proof. We prove this by induction on the number of vertices of H . For $n(H) = 1$ the collection of graphs satisfying the properties in the premise is empty, whence the assertion trivially holds. Therefore suppose that H satisfies the premises, $n(H) > 1$ and that the assertion holds for all graphs on fewer vertices than H .

Since $\nu(H) \leq 2$ we have that every trivalent vertex v in H has second valency at most eleven, since otherwise $\nu(H_v) \leq \nu(H) - 3 \leq -1$. Hence $\delta(H_3) \geq 1$.

Suppose that $v \in V(H_3)$ with $d(H_3; v) = 1$, then $d^2(H; v) = 11$. Note that $\nu(H_v) \leq \nu(H) + 0 = 2$ and therefore we have, by Property 22, that any bivalent vertex of H_v would lie in a C_5 -component. However, if there were a C_5 -component, C , in H_v then we would need to have that all five vertices of C is adjacent to some vertex in $N(v)$. This would however give a cycle of length four through some of the vertices in $N(v)$. Thus there are no bivalent vertices in H_v and therefore no trivalent vertices at distance two from v in H . Hence if $v \in V(H_3)$ is such that $d(H_3; v) = 1$ then all vertices at distance two from v are tetravalent. This means that v belongs to a K_2 -component of H_3 . \square

Lemma 16. If $H \leq G$ is such that $\delta(H) = 2$, $v \in V(H_2)$ and $\nu(H) \leq 5$ then one of the following holds

- (i) $d^2(H; v) \geq 5$.
- (ii) $v \in V(C)$ where $C_5 \cong C \in \mathcal{C}(H)$.
- (iii) $\exists e \in E(H) : v \in V(C)$ where $C_5 \cong C \in \mathcal{C}(H - e)$.

Proof. Let H and v be as in the premises and suppose that $d^2(H; v) \leq 4$ and that v does not lie in a C_5 -component of H . We then want to show that (iii) holds. By Property 7 H contains no 2-regular components and therefore neither does H_2 .

Since $\delta(H) = 2$ we get that $d^2(H; v) = 4$, whence v has two bivalent neighbours, u and w . If neither u nor w has a bivalent neighbour apart from v , then $\nu(H_u) \leq \nu(H) + 1 \leq 6$. Since $e(N(u), N(w)) \leq 1$ the monovalent vertex w has second valency at least two in H_u , whence $\nu(H_{u,w}) \leq \nu(H_u) - 7 \leq -1$, contradicting assumption (A1).

Hence, at least one of u and w has a bivalent neighbour and v belongs to a path component, P , of H_2 of length at least four. Let $a, b \in V(P)$ be the endpoints of the path component P . If $d^2(H; a) \geq 6$ then $\nu(H_a) \leq \nu(H) - 2 \leq 3$ but there would be a monovalent vertex (the vertex at distance two from a in P) in H_a , contradicting Property 4. Hence, $d^2(H; a) = 5$ and analogously we obtain $d^2(H; b) = 5$. Therefore $\nu(H_a) \leq \nu(H) + 1 \leq 6$ and the vertex at distance two from a in P must belong to a K_2 -component of H_a by Property 9. This is only possible if b is at distance three from a , in P , and the trivalent neighbours of a and b , say x , coincide. The edge $e = xy \in E(H)$ that is neither incident to a nor b is then such that (iii) holds. \square

In the remainder of this section we assume $H = G_v$ for some vertex $v \in V(G)$. In particular we then have that H_3 is a graph on 12 vertices since G is 4-regular by Property 31, and by Lemma 15 we have that the minimum valency in H_3 is one and every monovalent vertex belongs to some K_2 -component. Every vertex of H which is not trivalent is tetravalent.

Lemma 17. H_3 contains no cycles of length five.

Proof. Since G is 4-regular we have that each vertex of $V(H_3)$ is adjacent to one of the neighbours of v in G . If there were a cycle of length five in H_3 then since $|N_G(v)| = 4$ one of the neighbours of v would have to be adjacent to at least two vertices in the cycle, contradicting Lemma 13. \square

Note that Lemmas 15 and 17 corresponds to [9, Lemma 5.2.3].

Lemma 18. (Analogue of [9, Lemma 5.2.4]) H_3 contains no cycles of length six.

Proof. Since G is 4-regular we have that $\nu(H) \leq \nu(G) + 2 \leq 2$.

Suppose that $\{c_1, c_2, c_3, c_4, c_5, c_6\} \subseteq V(H_3)$ is a cycle of length six in H_3 . Suppose one of the c_i has a tetravalent neighbour in H , without loss of generality assume then that $d^2(H; c_1) = 10$. Then $\nu(H_{c_1}) \leq \nu(H) + 3 \leq 5$. If c_1 and c_4 have a common neighbour then, by Lemma 16, either there is an edge $e \in E(H_{c_1})$: $c_4 \in V(C)$, where $C_5 \cong C \in \mathcal{C}(H_{c_1} - e)$, or $c_4 \in V(C)$ where $C \cong C_5 \in \mathcal{C}(H_{c_1})$. But then the four or five, in H_{c_1} , bivalent vertices of $V(C)$ has to be adjacent to the three neighbours of c_1 . Accordingly there is a vertex with more than one neighbour in a cycle of length five, contradicting Lemma 13.

Hence, c_3 and c_5 are bivalent in H_{c_1} with at least one trivalent neighbour, c_4 . If c_3 has second valency at least six in H_{c_1} then $\nu(H_{c_1, c_3}) \leq \nu(H_{c_1}) - 2 \leq 3$, but c_5 is then monovalent in H_{c_1, c_3} contradicting Property 4. Hence $d^2(H_{c_1}; c_3) \leq 5$ and therefore $d(H_{c_1}; x_3) = 2$ where $x_3 \in N_H(c_3) \setminus \{c_2, c_4\}$. Since H contains no cycle of length four, x_3 has at most one common neighbour with c_1 , and therefore x_3 has to be trivalent in H . Similarly, if x_5 is such that $x_5 \in N_H(c_5) \setminus \{c_4, c_6\}$ then x_5 is trivalent in H and has a common neighbour with c_1 . Since $d^2(H_{c_1}; c_3) = 5$ we get $\nu(H_{c_1, c_3}) \leq \nu(H_{c_1}) + 1 \leq 6$ and since c_5 is monovalent in H_{c_1, c_3} we get, by Property

9, that x_5 also is monovalent in H_{c_1, c_3} . Hence, $x_3x_5 \in E(H)$ and therefore we have a cycle $c_3c_4c_5x_5x_3$ of length five in H_3 , contradicting Lemma 17.

Thus all of the vertices c_i of the 6-cycle have second valency nine. Clearly c_1 and c_4 have no common neighbour or we would get a cycle of length five in H_3 , contradicting Lemma 17. Hence $\nu(H_{c_1}) \leq \nu(H) + 6 \leq 8$ and c_3 is bivalent in H_{c_1} with $d^2(H_{c_1}; c_3) \geq 6$ (since c_1 and c_3 do not have adjacent neighbours or we would get a cycle of length five or less in H_3), whence $\nu(H_{c_1, c_3}) \leq \nu(H_{c_1}) - 2 \leq 6$. But $e(N(c_1) \cup N(c_3), N(c_5)) = 0$ or we would get a cycle of length five or less in H_3 . Therefore c_5 is monovalent in H_{c_1, c_3} but has second valency three, giving $\nu(H_{c_1, c_3, c_5}) \leq \nu(H_{c_1, c_3}) - 8 \leq -2$, contradicting assumption (A1). \square

Lemma 19. (Analogue of [9, Lemma 5.2.5]) Let $x, y \in V(H_3)$ be bivalent in H_3 . Furthermore suppose $N_H(x) = \{t, x_1, x_2\}$ and $N_H(y) = \{t, y_1, y_2\}$ where $t, x_1, y_1 \in V(H_3)$. Then $x_1y_2, y_1x_2 \in E(H)$.

Proof. We will show that $e(N(x), N(y)) \geq 2$, since then $e(N(x), N(y)) = 2$, or we would get cycles of length four or less. Since $x_1y_1 \notin E(H)$, by Lemma 17, the only possibility is then that $x_1y_2, y_1x_2 \in E(N(x), N(y))$.

By Lemma 15 both x_1 and y_1 has another neighbour in H_3 except for x and y , let those be $x_3 \in V(H_3)$ and $y_3 \in V(H_3)$, respectively. Note that x_3 and y_3 are distinct by Lemma 18.

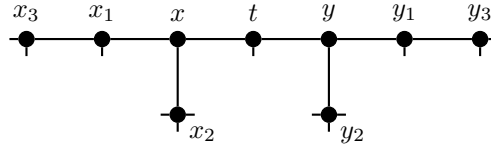


FIGURE 26. The neighbourhood of x and y in H .

If $e(N(x), N(y)) = 0$, then $\nu(H_{x,y}) \leq 0$ with x_3 and y_3 bivalent. Thus, x_3 and y_3 belong to a C_5 -component of $H_{x,y}$. Each of the five vertices of $N(x) \cup N(y)$ has a single neighbour in the C_5 -component, which would give a cycle of length five in H_3 , contradicting Lemma 17.

If $e(N(x), N(y)) = 1$ then we get $d^2(H_{x,y}) = 6$ and therefore $\nu(H_{x,y}) \leq \nu(H) + 3 - 2 \leq 3$. Note that x_3 and y_3 are bivalent in $H_{x,y}$. If $\alpha((H_{x,y})_2) > 1$, then (by Properties 22 and 28) there is a component $C \in \mathcal{C}(H_{x,y})$ such that either $C \cong C_5$ or there is an $e \in E(C)$ with the property that $C - e = C_5 + C'$, where $C' \in \{C_5, W_5, (2C_7)_{2i}\}$. By the same reasoning as we used for excluding the $e(N(x), N(y)) = 0$ -case we can see that $C \not\cong C_5$. In the other case, all of the vertices of the C_5 in $C - e$ must be adjacent to one (and exactly one) of the vertices in $N(x) \cup N(y)$, and vice versa. The reason for this is that we have no bivalent vertices in H and no cycles of length five in H_3 . There are six additional edges incident to $N(x) \cup N(y)$. The vertices $x_3, x_1, x, t, y, y_1, y_3$ and two more vertices of the C_5 -part of $C - e$ are trivalent in H . Thus, $|V(C') \cap V(H_3)| \leq 3$, since we have in total twelve trivalent vertices, and the nine listed previously are not in $V(C')$. But $|V((C')_3)| = 14$, so $|V(C_3)| = 14$ and therefore $|V(C_3) \cap V(C')| = 13$. Clearly it is then impossible to have $|V(C') \cap V(H_3)| \leq 3$, which is a contradiction.

Thus, $\alpha((H_{x,y})_2) = 1$. Therefore $y_3x_3 \in E(H)$ and y_3, x_3 are the only two bivalent vertices of $H_{x,y}$. We have also $d^2(x_1), d^2(y_1) \geq 10$. Observe that t and y_3 are non-adjacent bivalent vertices in H_{x_1} . This would however contradict Corollary 3.

Hence, $e(N(x), N(y)) \geq 2$, as desired. \square

Lemma 20. (Analogue of [9, Lemma 5.2.6]) If $t \in V(H)$ is trivalent in H_3 then it has two bivalent and one trivalent neighbour in H_3 .

Proof. Recall that the number of trivalent vertices in H is 12.

That $t \in V(H)$ is trivalent in H_3 means that $d_H^2(t) = 3 + 3 + 3$. Let $W = \{w_1, w_2, w_3\} = N(t)$ and suppose that at least two of the vertices w_1, w_2, w_3 have second valency nine. Then at least five out of the six distinct vertices at distance two from t are trivalent in H , and each has at least one other trivalent neighbour not in W . Since there are no cycles of length five or six in H_3 by Lemmas 17 and 18 there is then at least five trivalent vertices at distance three from t .

Hence, there is one trivalent at distance 0 from t , three at distance 1, at least five at distance 2 and at least five at distance 3, whence there are at least 14 trivalent vertices in H , contradicting that there are 12 such vertices.

Therefore, at most one of the vertices w_1, w_2 and w_3 has second valency nine. If none of them had second valency nine then each of them has a trivalent and a tetravalent neighbour except for t . Let x_1 be the trivalent neighbour of w_1 that is not t , moreover let y_2 and y_3 be the tetravalent neighbours of w_2 and w_3 , respectively. By applying Lemma 19 (for $(x, y) = (w_1, w_2), (w_1, w_3)$) we get that $x_1 y_2, x_1 y_3 \in E(H)$. But then w_1 is the only trivalent neighbour of x_1 , contradicting Lemma 15.

Hence, exactly one of the vertices w_1, w_2 and w_3 has second valency nine which completes the proof. \square

We define two graphs on twelve vertices, S_1 and S_2 , just as in [9].

Definition 3. Denote the vertices of C_{12} by $\{c_0, c_1, \dots, c_{11}\}$ so that $c_0 c_1 \dots c_{11}$ forms the cycle of length 12. The graph S_1 is formed by adding the edge $c_0 c_6$ to C_{12} and the graph S_2 is formed by adding the two edges $c_0 c_6$ and $c_3 c_9$ to the C_{12} .

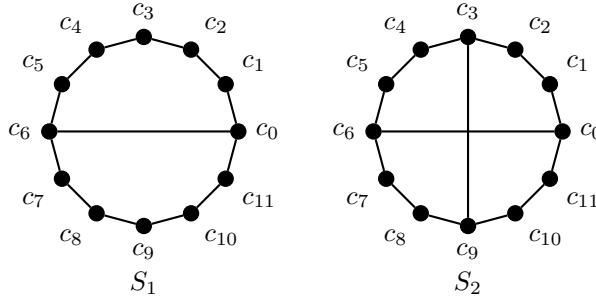


FIGURE 27. The two graphs S_1 and S_2 .

Lemma 21. (Analogue of [9, Lemma 5.2.7]) If $C \in \mathcal{C}(H_3)$, then $C \in \{K_2, C_8, C_{10}, C_{12}, S_1, S_2\}$.

Proof. Recall that $n(H_3) = 12$. If $C \in \mathcal{C}(H_3)$ then $\delta(C) \geq 2$ unless $C = K_2$ by Lemma 15. Suppose therefore that $\delta(C) \geq 2$, then $n(C) \geq 7$ because C contains no cycles of length five or six, by Lemmas 17 and 18. Since the only possible component in H_3 with no more than six vertices is K_2 we get that C has an even number of vertices.

Then either C is 2-regular and then a cycle of length eight, ten or twelve, or it has at least one vertex of valency three. Suppose therefore that C has a trivalent vertex, then it has at least two, and every trivalent vertex is paired with one other adjacent trivalent vertex, by Lemma 20.

If there are two trivalent vertices, then the only possible graph is $C = S_1$ and if there are four trivalent vertices the only possibility is $C = S_2$. It is not possible to have more than four trivalent vertices in H_3 . \square

Lemma 22. (Analogue of [9, Lemma 5.2.8]) Let $C \in \mathcal{C}(H_3)$, then $C \notin \{C_8, C_{10}, C_{12}\}$.

Proof. Suppose that $C \in \mathcal{C}(H_3)$ and $C = C_k$ where $k \in \{8, 10, 12\}$. Let the vertices of C_k be cyclically labelled by c_1, c_2, \dots, c_k . By Lemma 19 we have that two vertices at distance three in the cycle must have a common tetravalent neighbour in H .

If $k = 8$ then c_1 and c_4 have a common tetravalent neighbour, but so does c_4 and c_7 . Since the vertices of $V(C)$ only have one tetravalent neighbour each we then get that all three vertices c_1, c_4 and c_7 have a common tetravalent neighbour, contradicting Lemma 13.

If $k = 10$ then the following pairs of vertices have common tetravalent neighbours: $(c_1, c_4), (c_4, c_7)$ and (c_7, c_{10}) . But then c_1, c_4, c_7, c_{10} would all be adjacent to the same tetravalent vertex, again contradicting Lemma 13.

Finally, if $k = 12$ then we get in the same manner as above that c_i, c_{i+3}, c_{i+6} and c_{i+9} have a common tetravalent neighbour, v_i , for $i \in \{1, 2, 3\}$. Since each of the three tetravalent vertices v_1, v_2 and v_3 have four neighbours in $V(C)$ they form a component C' of H . However, C' is a graph with 24 edges, 15 vertices and independence number 6 and therefore, $\nu(C') = 27$, contradicting $\nu(H) \leq 2$ since $H - C' < G$ and G is assumed to be a minimal counterexample to Theorem 1. \square

Lemma 23. (Analogue of [9, Lemma 5.2.9]) Let $C \in \mathcal{C}(H_3)$, then $C \notin \{S_1, S_2\}$.

Proof. For $C \in \{S_1, S_2\}$, then by applying Lemma 19 repeatedly we get that c_i, c_{i+3}, c_{i+6} and c_{i+9} have a common tetravalent neighbour, v_i , for $i \in \{1, 2\}$. If $C = S_2$, then v_1, v_2 and $V(C)$ form a component C' of H . This component would have 14 vertices, 22 edges and independence number 5, whence $\nu(C') = 3$, contradicting that $\nu(H) \leq 2$ and $\nu(H - C') \geq 0$.

If $C = S_1$, then c_3 and c_9 have a (possibly common) tetravalent neighbour that is not v_1 or v_2 . Let $S = \{c_1, c_3, c_5, c_7, c_9, c_{11}\}$. If $N(c_3) \cap N(c_9) = \emptyset$, then $n(H_S) = n(H) - 16$, $e(H_S) = e(H) - 29$ and $\alpha(H_S) \leq \alpha(H) - 6$. Hence $\nu(H_S) \leq \nu(H) - 25 \leq -23$, contradicting assumption (A1). On the other hand if $N(c_3) \cap N(c_9) \neq \emptyset$ then $n(H_S) = n(H) - 15$, $e(H_S) = e(H) - 25$ and $\alpha(H_S) \leq \alpha(H) - 6$, whence $\nu(H_S) \leq \nu(H) - 30$ also contradicting assumption (A1). \square

Note that by Lemmas 21, 22 and 23 we get that $H_3 \cong 6K_2$, since H_3 has twelve vertices.

Lemma 24. G contains no cycles of length six.

Proof. Suppose that the vertices $\{c_1, c_2, \dots, c_6\} \subseteq V(G)$ formed a cycle of length six (labelled cyclically in order). Let $H = G_{c_1}$, then since $H_3 \cong 6K_2$ we get that $d(H_3; c_3) = 1$. However, c_5 would also be trivalent in H and at distance two from c_3 , contradicting Lemma 15. \square

Lemma 25. Through every pair of incident edges in G there is a cycle of length five.

Proof. Suppose otherwise and let $ux, xv \in E(G)$ be a pair of incident edges through which there is no cycle of length five. We have $d^2(u) = 16$ and therefore $\nu(G_u) \leq \nu(G) + 2 \leq 2$. Since there is no cycle of length five through u, x, v we have that $e(N(u), N(v)) = 0$ and therefore $d^2(G_u; v) = 12$. This would however give $\nu(G_{u,v}) \leq \nu(G_v) - 3 \leq -1$, contradicting assumption (A1). \square

Lemma 26. Two cycles of length five in G share at most one edge.

Proof. Suppose otherwise, then the two cycles, C_1 and C_2 , of length five would share exactly two consecutive edges or we would get a cycle of length four or less in G . Suppose therefore that the edges shared are $x_1v \in E(G)$ and $vx_2 \in E(G)$. Let x_3 and x_4 be the two remaining neighbours of v , $H = G_v$ and $X_i = N(x_i) \setminus \{v\}$. Because of Lemma 25 there are also cycles of length five through the pairs of incident edges (x_2v, vx_3) and (x_2v, vx_4) , whence $|E(X_2, X_3)|, |E(X_2, X_4)| \geq 1$. But since two of the vertices in X_2 belong to C_1 or C_2 , and as such get paired with a vertex in X_1 we must have that there are edges from X_3 and X_4 to the same vertex in X_2 , which then is bivalent in H_3 , contradicting that $H_3 \cong 6K_2$. \square

Now we are ready to prove a lemma that will contradict Lemma 24 and complete the proof of the main theorem (Theorem 1).

Lemma 27. (Analogous of an argument in the proof of [2, Theorem 3]) G contains at least one cycle of length six.

Proof. Let $uv \in E(G)$. By Lemmas 25 and 26 there are two cycles C, C' of length five through uv which do not share any other edge than uv . Let $S = \{x_1, x_2, x_3, x_4\} = N(\{u, v\}) \cap (V(C) \cup V(C'))$ where x_1, x_2 are adjacent to u and x_3, x_4 are adjacent to v . If the only edges between the neighbourhoods of the x_i were the edges in $E(N(x_1), N(x_2))$ and $E(N(x_3), N(x_4))$ guaranteed by Lemma 25 then we would get $\nu(G_S) \leq \nu(G) + 2 + 0 + 0 - 5 = -3$, contradicting assumption (A1). If $e(N(x_i), N(x_{i+1})) \geq 2$, where $i \in \{1, 3\}$, then we either get a cycle of length four or a cycle of length six through x_i and x_{i+1} .

If there are no cycle of length six through the x_1, x_2 or x_3, x_4 we must instead have an edge in $E(N(x_1) \cup N(x_2) \setminus \{u\}, N(x_3) \cup N(x_4) \setminus \{v\})$ which gives a cycle of length six through uv . \square

3.2. Proof of main theorem. We have now shown that if G is a minimal counterexample, either to ν being non-negative or to G being such that $G \notin \mathcal{G}$ but with ν -value zero, then G must contain cycles of length six by Lemma 27 but also not contain cycles of length six by Lemma 24. Hence, there is no minimal counterexample and therefore no counterexample at all. We have therefore shown Theorem 1 to hold. We know that all triangle-free graphs have non-negative ν -value and we that those that have ν -value zero are exactly those in $\{W_5, (2C_7)_{2i}\} \cup \{Ch_k; k \geq 2\} \cup \{BC_k; k \geq 5\}$.

Now, since Theorem 1 are proven, all the properties of Section 2 that were made under assumption (A1) now can be read as general propositions. For example we have fully classified the graphs with ν -value at most two with bivalent vertices (see Property 22).

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